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Split Orthodox Semigroups

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Let S be an orthodox semigroup (that is, a regular semigroup whose idempotents form a subsemigroup) and let $\mathscr{Y}$ be the finest inverse semigroup congruence on S. In this paper we consider those orthodox semigroups S for which the natural homomorphism $\natural: S \to S/\mathscr{Y}$ is *split* in that there is a homomorphism $\pi: S/\mathscr{Y} \to S$ such that $\pi\natural = id_{S/\mathscr{Y}}$. Our objective being to obtain a structure theorem for such semigroups, we devote section 1 to preparing the way for this and indicate some situations where splitting occurs. In section 2 we present the main structure theorem for split orthodox semigroups. In order to derive an analogous result for ordered orthodox semigroups we consider, in Section 3, a particular type of equivalence relation on an ordered set that arises naturally in connection with splitting and list some properties of $\mathscr{Y}$ relative to Green's relations on the band of idempotents. In Section 4 we obtain the ordered analogue of the main structure theorem. We then consider, in Section 5, the presence of a greatest idempotent and show that in this case splitting occurs under very reasonable conditions; in particular, when $\mathscr{Y}$ is a multiplicative closure equivalence.

For the reader's convenience, we recall that on an orthodox semigroup S the relation $\mathscr{Y}$ is given by $a \mathscr{Y} b \Leftrightarrow V(a) = V(b)$ where $V(a)$ denotes the set of inverses of a [4]. Moreover, $\mathscr{Y}$ is idempotent-determined in that if B is the band of idempotents of S and $b \in B$ then $a \mathscr{Y} b$ implies $a \in B$. On B, the $\mathscr{Y}$-classes are the same as the $\mathscr{D}$-classes; and the reader is advised that throughout this paper the letter $\mathscr{D}$ will always denote Green's relation on B. The $\mathscr{D}$-classes are rectangular bands (so that if x, y, z are $\mathscr{D}$-equivalent idempotents then $xyz = xz$;

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and if x, y commute then $x = y$). We shall also have occasion to use the fact that $\mathscr{Y} \cap \mathscr{H}$ is the identity relation on S.

1. Splitting and Skeletons

Definition 1.1. Let $B = \bigcup\{B_\alpha; \alpha \in Y\}$ be a band with structure semilattice Y and $\mathscr{D}$-classes the rectangular bands B_α. If $\natural: B \to B/\mathscr{D}$ is the natural homomorphism then we shall say that B is *split* if there is a homomorphism $\pi: B/\mathscr{D} \to B$ such that $\pi\natural = id_{B/\mathscr{D}}$. Such a homomorphism π will be called a *splitting homomorphism.*

If a splitting homomorphism π exists then it is clear that B contains a subsemilattice (= commutative sub-band) that meets every $\mathscr{D}$-class of B exactly once. We thus consider also the following notion.

Definition 1.2. Let $B = \bigcup\{B_\alpha; \alpha \in Y\}$ be a band. Then by a *skeleton* of B we shall mean a subset $E = \{x_\alpha; \alpha \in Y\}$ such that $x_\alpha \in B_\alpha$ for every $\alpha \in Y$ and $x_\alpha x_\beta = x_{\alpha\beta} = x_\beta x_\alpha$ for all $\alpha, \beta \in Y$.

Lemma 1.3. *A band B is split if and only if it has a skeleton. If* $\pi: B/\mathscr{D} \to B$ *is a splitting homomorphism then* $\operatorname{Im} \pi$ *is a skeleton of B.*

We can clearly extend the above notion of splitting to a general orthodox semigroup.

Definition 1.4. Let T be an orthodox semigroup and let $\natural: T \to T/\mathscr{Y}$ be the natural homomorphism. Then we shall say that T is *split* if there is a homomorphism $\pi: T/\mathscr{Y} \to T$ such that $\pi\natural = id_p/_{\mathscr{Y}}$.

It is clear that if T splits then so does its band B of idempotents. Moreover, T splits if and only if it contains an inverse subsemigroup that meets every $\mathscr{Y}$-class exactly once. In order to investigate this situation more closely, we consider the following notion.

Definition 1.5. Let T be an orthodox semigroup with band of idempotents B. Suppose that E is a $\mathscr{D}$-transversal of B in that E meets every $\mathscr{D}$-class exactly once. Then we define the *span* of E by

$$\operatorname{Sp}(E) = \{a \in T; (\exists e, f \in E)\, e \mathscr{R} a \mathscr{L} f\}.$$

Lemma 1.6. $\operatorname{Sp}(E)$ *meets every* $\mathscr{Y}$*-class of* T *exactly once.*

Proof. Given $a \in T$, let a' be an inverse of a; then $aa' \mathscr{R} a \mathscr{L} a'a$. Now since E is a $\mathscr{D}$-transversal of B there exist unique $e, f \in E$ such that $e \mathscr{D} aa'$ and

$f \mathscr{D} a'a$. Writing $eaf = \tilde{a}$, we therefore have $\tilde{a} \mathscr{Y} aa'aa'a = a$. Now let $a^0 = fa'e$; then

$$\begin{aligned}\tilde{a}a^0\tilde{a} &= eaf \cdot fa'e \cdot eaf\\ &= ea \cdot a'afa'a \cdot a' \cdot aa'eaa' \cdot af\\ &= ea \cdot a'a \cdot a' \cdot aa' \cdot af \qquad \text{since } f \mathscr{D} a'a,\ e \mathscr{D} aa'\\ &= eaf\\ &= \tilde{a}\end{aligned}$$

and similarly $a^0 = a^0\tilde{a}a^0$. It now follows that

$$\begin{aligned}\tilde{a} \mathscr{R} \tilde{a}a^0 &= eaf \cdot fa'e\\ &= ea \cdot a'afa'a \cdot a'e\\ &= ea \cdot a'a \cdot a'e \qquad \text{since } f \mathscr{D} a'a\\ &= eaa'e\\ &= e \qquad\qquad \text{since } e \mathscr{D} aa'\end{aligned}$$

and similarly that $\tilde{a} \mathscr{L} a^0\tilde{a} = f$. Consequently we have $\tilde{a} \in \mathrm{Sp}(E)$, so that $\mathrm{Sp}(E)$ meets every $\mathscr{Y}$-class at least once.

Note that this situation may be summarised by the scheme

$$\begin{array}{ccccc} e & \mathscr{R} & \tilde{a} & \mathscr{L} & f\\ \mathscr{D} & & \mathscr{Y} & & \mathscr{D}\\ aa' & \mathscr{R} & a & \mathscr{L} & a'a. \end{array}$$

To show that the span of E meets every $\mathscr{Y}$-class precisely once, suppose that $a, b \in T$ are such that $a \mathscr{Y} b$ with $b \in \mathrm{Sp}(E)$. Consider the following scheme in which $b' \in V(b)$ and $u, v \in E$:

$$\begin{array}{ccccc} e & \mathscr{R} & \tilde{a} & \mathscr{L} & f\\ \mathscr{D} & & \mathscr{Y} & & \mathscr{D}\\ aa' & \mathscr{R} & a & \mathscr{L} & a'a\\ * & & \mathscr{Y} & & *\\ bb' & \mathscr{R} & b & \mathscr{L} & b'b\\ & & \Vert & & \\ u & \mathscr{R} & b & \mathscr{L} & v. \end{array}$$

Since $aa' \mathscr{Y} ba' \mathscr{Y} bb'$ and $a'a \mathscr{Y} a'b \mathscr{Y} b'b$ we can replace each * in the above by $\mathscr{D}$. Consequently we see that $e \mathscr{D} u$ and $f \mathscr{D} v$ whence $e = u$ and $f = v$ (since

$e, u, f, v \in E$ and E is a $\mathscr{D}$-transversal of B). It now follows that $\tilde{a}\,\mathscr{R}\,e = u\,\mathscr{R}\,b$ and $\tilde{a}\,\mathscr{L}\,f = v\,\mathscr{L}\,b$ so that $(\tilde{a}, b) \in \mathscr{Y} \cap \mathscr{H}$. We conclude that $\tilde{a} = b$ so that $\mathrm{Sp}(E)$ meets every $\mathscr{Y}$-class exactly once. ∎

Note from the above that if $a \in \mathrm{Sp}(E)$ with $e\,\mathscr{R}\,a\,\mathscr{L}\,f$ where $e, f \in E$ then a^0 is an inverse of a such that $f\,\mathscr{R}\,a^0\,\mathscr{L}\,e$. Thus $a^0 \in \mathrm{Sp}(E)$ and it follows immediately from Lemma 1.6 that a^0 is the unique inverse of a in $\mathrm{Sp}(E)$; we shall denote it henceforth by a^{-1}. Note also that $e, f \in E$ are uniquely determined by a; indeed, $e = aa^{-1}$ and $f = a^{-1}a$.

THEOREM 1.7. *Let T be an orthodox semigroup with band of idempotents B and suppose that B has a skeleton E. Then the following statements are equivalent*:

(1) *there is an inverse subsemigroup S of T that meets every $\mathscr{Y}$-class of T exactly once and has E as semilattice of idempotents*;

(2) $aEa^{-1} \subseteq E$ *for every* $a \in \mathrm{Sp}(E)$;

(3) $\mathrm{Sp}(E)$ *is a subsemigroup of* T.

Moreover, if (1) *holds then necessarily* $S = \mathrm{Sp}(E)$.

Proof. (1) $\Rightarrow$ (2): Given $a \in S$ we have $e\,\mathscr{R}\,a\,\mathscr{L}\,f$ for some idempotents e, f in S. Since the semilattice of idempotents of S is E, it follows that $a \in \mathrm{Sp}(E)$, whence $S \subseteq \mathrm{Sp}(E)$. Now both S and $\mathrm{Sp}(E)$ meet every $\mathscr{Y}$-class exactly once and, by Lemma 1.6, if $a \in S$ then $\tilde{a} \in \mathrm{Sp}(E)$ with $\tilde{a}\,\mathscr{Y}\,a$. Hence we see that $S = \mathrm{Sp}(E)$. Since E is the semilattice of idempotants of $S = \mathrm{Sp}(E)$, (2) now follows.

(2) $\Rightarrow$ (3): Given $a, b \in \mathrm{Sp}(E)$ we have $a^{-1}, b^{-1} \in \mathrm{Sp}(E)$ and $a^{-1}a, bb^{-1} \in E$. Since then $abb^{-1}a^{-1} \cdot ab = aa^{-1}abb^{-1}b = ab$ and similarly $ab \cdot b^{-1}ab = aa^{-1}abb^{-1}b = ab$ we see that

$$abb^{-1}a^{-1}\,\mathscr{R}\,ab\,\mathscr{L}\,b^{-1}a^{-1}ab.$$

Since, by (2), $abb^{-1}a^{-1} \in aEa^{-1} \subseteq E$ and $b^{-1}a^{-1}ab \in b^{-1}Eb \subseteq E$ it follows that $\mathrm{Sp}(E)$ is a subsemigroup of T.

(3) $\Rightarrow$ (1): If $\mathrm{Sp}(E)$ is a subsemigroup then, by the remarks following Lemma 1.6, $\mathrm{Sp}(E)$ is an inverse semigroup which, moreover, meets very $\mathscr{Y}$-class exactly once and has E as its set of idempotents. ∎

Before considering the structure of split orthodox semigroups we consider briefly the case where the band B of idempotents of T is *normal* in that the identity $efgh = egfh$ holds.

THEOREM 1.8. *Let T be an orthodox semigroup whose band B of idempotents is normal. Then T is split if and only if B is split.*

Proof. If T is split then clearly so is B. Conversely, suppose that B is split, so that B has a skeleton E. We show as follows that $\mathrm{Sp}(E)$ is a subsemigroup of T, whence the result follows by Theorem 1.7.

Let $a, b \in \mathrm{Sp}(E)$ with, say, $e \mathscr{R} a \mathscr{L} f$ and $u \mathscr{R} b \mathscr{L} v$ where $e, f, u, v \in E$. Then there exist $e_{ab}, f_{ab} \in E$ such that

$$e_{ab} \mathscr{D} abb^{-1}a^{-1} \qquad \text{and} \qquad f_{ab} \mathscr{D} b^{-1}a^{-1}ab.$$

Now $e = aa^{-1}$ and so we have

$$ee_{ab}e \mathscr{D} eabb^{-1}a^{-1}e = abb^{-1}a^{-1} \mathscr{D} e_{ab}$$

whence $ee_{ab}e = e_{ab}e$ and e_{ab} belong to the skeleton E. Hence

$$\begin{aligned} e_{ab} \cdot abb^{-1}a^{-1} &= e \cdot ee_{ab}e \cdot eabb^{-1}a^{-1}e \cdot e \\ &= e \cdot eabb^{-1}a^{-1}e \cdot ee_{ab}e \cdot e \qquad \text{since } B \text{ is normal} \\ &= abb^{-1}a^{-1} \cdot e_{ab} \end{aligned}$$

and so e_{ab} and $abb^{-1}a^{-1}$ commute. But these elements are $\mathscr{D}$-equivalent in B; so $e_{ab} = abb^{-1}a^{-1}$. Similarly we can show that $f_{ab} = b^{-1}a^{-1}ab$. Consequently we see that $ab \in \mathrm{Sp}(E)$ and so $\mathrm{Sp}(E)$ is a subsemigroup. ∎

COROLLARY 1.9. *Let T be an orthodox semigroup whose band B of idempotents is normal. Suppose that $T/\mathscr{Y}$ has an identity. Then T is split.*

Proof. Let $\xi \in B$ be such that $\mathscr{Y}_\xi$ is the identity of $T/\mathscr{Y}$. Then for every $a \in B$ we have $a \mathscr{D} \xi a \xi$. Now let

$$E = \{\xi a \xi;\, a \in B\}.$$

By the above observation, E meets every $\mathscr{D}$-class of B. Now $\xi \in B$ so for all $a, b \in B$ we have

$$\begin{aligned} \xi a \xi \cdot \xi b \xi &= \xi \cdot \xi a \xi \cdot \xi b \xi \cdot \xi \\ &= \xi \cdot \xi b \xi \cdot \xi a \xi \cdot \xi \qquad \text{since } B \text{ is normal} \\ &= \xi b \xi \cdot \xi a \xi, \end{aligned}$$

whence E is a subsemilattice of B. Finally, if $\xi a \xi \mathscr{D} \xi b \xi$ then, since E is commutative and every $\mathscr{D}$-class of B is a rectangular band, we have $\xi a \xi = \xi b \xi$. Thus E meets every $\mathscr{D}$-class of B exactly once. B is therefore split whence, by Theorem 1.8, so is T. ∎

Another situation where splitting occurs is the following.

THEOREM 1.10. *If T is an orthodox semigroup such that $T/\mathscr{Y}$ is bicyclic then T is split.*

Proof. Let $T/\mathscr{Y} = B(p, q)$ with $pq = 1$, $qp \neq 1$. Choose $e, c \in T$ such that $e^2 = e$ and $e\natural = 1$ and $c\natural = p$. Consider the elements $a = ece$ and $b = ea'e$ where $a' \in V(a)$. Clearly, we have $a\natural = p$ and $b\natural = a'\natural = q$. Moreover, $(aa')\natural = a\natural a'\natural = pq = 1$ so that $aa' \mathscr{D} e$. It follows that

$$ab = aea'e = aa'e = eaa'e = e.$$

Also, since $(ba)\natural = b\natural a\natural = qp \neq 1$ we have $ba \neq e$. Consequently, $\langle a, b\rangle$ is bicyclic and so there is an isomorphism $\pi^+ \colon B(p, q) \to \langle a, b\rangle$, namely that given by $p\pi^+ = a$, $q\pi^+ = b$. Clearly, π^+ induces a homomorphism $\pi \colon B(p, q) \to T$, namely that given by $x\pi = x\pi^+$. Since $\pi\natural = id_{B(p,q)}$ it follows that π is a splitting homomorphism. ∎

T. E. Hall [5] defines a band B to be *almost commutative* if, given $e, f \in B$, either $ef = fe$ or $e \mathscr{D} f$. It is readily seen that any almost commutative band has a skeleton; namely, any $\mathscr{D}$-transversal.

2. Structure Theorems

Suppose now that B is a band and that E is a skeleton of B. We shall denote by T_B the set of band isomorphisms θ between subbands of B of the form eBe, where $e \in E$, such that θ is *skeleton-preserving* in that

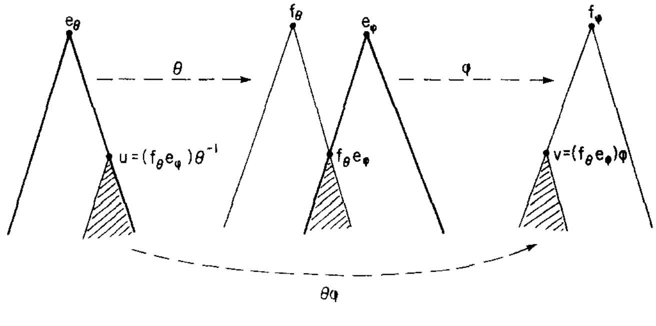
$$(\forall f \in \operatorname{Dom} \theta) \qquad f\theta \in E \Leftrightarrow f \in E.$$

It is clear that T_B is a generalisation of the corresponding notion T_E introduced by Munn [6]. We define composition of elements of T_B in the following way: given $\theta, \phi \in T_B$ with, say, $\theta\colon e_\theta B e_\theta \to f_\theta B f_\theta$ and $\phi\colon e_\phi B e_\phi \to f_\phi B f_\phi$ we define $\theta\phi$ to have domain uBu where $u = (f_\theta e_\phi)\theta^{-1} \in E$ and codomain vBv where $v = (f_\theta e_\phi)\phi \in E$; and $x(\theta\phi) = (x\theta)\phi$ for all $x \in \operatorname{Dom} \theta\phi$.

Now the natural order on B is given by

$$f \leqslant e \Leftrightarrow fe = ef = f \Leftrightarrow f = efe \Leftrightarrow f \in eBe.$$

Relative to this, the above definition of $\theta\phi$ has the following pictorial representation:



With the above notation, we thus see that

$$e_{\theta\phi} = u = (f_\theta e_\phi)\theta^{-1} \qquad \text{and} \qquad f_{\theta\phi} = v = (f_\theta e_\phi)\phi.$$

LEMMA 2.1. *T_B is an inverse semigroup.*

Proof. If $\theta, \phi \in T_B$ and $f \in \operatorname{Dom} \theta\phi$ then

$$f\theta\phi \in E \Leftrightarrow f\theta \in E \Leftrightarrow f \in E$$

so $\theta\phi \in T_B$, whence T_B is a semigroup. Being clearly closed under inverses, T_B is then an inverse semigroup. ▮

We can extend each $\theta \in T_B$ to a mapping $\bar{\theta}$: $B \to B$ by defining

$$(\forall b \in B) \qquad b\bar{\theta} = (e_\theta b e_\theta)\theta.$$

We then have the following property.

LEMMA 2.2. $(\forall \theta, \phi \in T_B)\ \overline{\theta\phi} = \bar{\theta}\bar{\phi}$.

Proof. For every $b \in B$ we have

$$\begin{aligned}
b\overline{\theta\phi} &= (e_{\theta\phi} b e_{\theta\phi})\,\theta\phi \\
&= (e_{\theta\phi} e_\theta b e_\theta e_{\theta\phi})\,\theta\phi \\
&= (e_{\theta\phi}\theta \cdot (e_\theta b e_\theta)\theta \cdot e_{\theta\phi}\theta)\phi \\
&= (e_\phi f_\theta \cdot (e_\theta b e_\theta)\theta \cdot f_\theta e_\phi)\phi && \text{since} \quad e_{\theta\phi}\theta = f_\theta e_\phi = e_\phi f_\theta \\
&= (f_\theta \cdot (e_\theta b e_\theta)\theta \cdot f_\theta)\bar{\phi} \\
&= ((e_\theta b e_\theta)\theta)\bar{\phi} && \text{since} \quad f_\theta = e_\theta\theta \\
&= (b\bar{\theta})\bar{\phi}. \ ▮
\end{aligned}$$

Given $\theta \in T_B$, say θ: $e_\theta B e_\theta \to f_\theta B f_\theta$, it is clear that $E \cap \operatorname{Dom} \theta = e_\theta E$ and $E \cap \operatorname{Cod} \theta = f_\theta E$ and that θ induces an isomorphism $\hat{\theta}$: $e_\theta E \to f_\theta E$; for every $x \in e_\theta E$ we have $x\hat{\theta} = x\theta$. The assignment $\hat{}: \theta \mapsto \hat{\theta}$ is then a homomorphism from T_B to T_E, the semigroup of isomorphisms between principal ideals of E.

Suppose now that S is an inverse semigroup with semilattice of idempotents E and consider the diagram

$$\begin{tikzcd}
S \arrow[dr, "\mu"'] & T_B \arrow[d, "\hat{}"] \\
& T_E
\end{tikzcd}$$

in which $\mu: S \to T_S$, described by $\mu: a \mapsto \mu_a$, is the Munn homomorphism, so that $e\mu_a = a^{-1}ea$ for every $e \in \operatorname{Dom} \mu_a = aa^{-1}E$.

A homomorphism $\theta: S \to T_B$ making this diagram commutative will be called a *triangulation* of the Munn homomorphism μ. Such a triangulation is necessarily idempotent-separating.

Concerning this situation, we have the following technical result that is fundamental to the discussion that follows.

LEMMA 2.3. *Let S be an inverse semigroup with semilattice of idempotents E and let B be a band with skeleton E. For every $a \in S$ let the domain and codomain of μ_a be e_aE (so that $e_a = aa^{-1}$) and f_aE (so that $f_a = a^{-1}a$). Let θ be a triangulation of μ. Then given $a, b \in S$ and $e, f, u, v \in B$ such that $e \mathscr{L} e_a$, $f \mathscr{R} f_a$, $u \mathscr{L} e_b$, $v \mathscr{R} f_b$ we have*

$$e \cdot (fu)\,\bar{\theta}_{a^{-1}} \mathscr{L} e_{ab} \qquad \textit{and} \qquad (fu)\,\bar{\theta}_b \cdot v \mathscr{R} f_{ab}\,.$$

Proof. On the one hand we have

$$\begin{aligned}
e(fu)\,\bar{\theta}_{a^{-1}} \cdot e_{ab} &= e(fu)\,\bar{\theta}_{a^{-1}}(f_ae_b)\,\mu_{a^{-1}} \\
&= e(fu)\,\bar{\theta}_{a^{-1}}(f_ae_b)\,\hat{\theta}_{a^{-1}} \\
&= e(fu)\,\bar{\theta}_{a^{-1}}(f_ae_b)\,\theta_{a^{-1}} \qquad \text{since} \quad f_ae_b \in E \\
&= e(f_{a^{-1}}fuf_a)\,\theta_{a^{-1}}(f_ae_b)\,\theta_{a^{-1}} \\
&= e(f_afue_bf_a)\,\theta_{a^{-1}} \qquad \text{since} \quad f_ae_b = e_bf_a \\
&= e(f_afuf_a)\,\theta_{a^{-1}} \qquad \text{since} \quad ue_b = u \\
&= e(fu)\,\bar{\theta}_{a^{-1}}\,;
\end{aligned}$$

and on the other hand

$$\begin{aligned}
e_{ab} \cdot e(fu)\,\bar{\theta}_{a^{-1}} &= e_{ab}e_ae(fu)\,\bar{\theta}_{a^{-1}} \\
&= e_{ab}(fu)\,\bar{\theta}_{a^{-1}} \\
&= (f_ae_b)\,\theta_{a^{-1}}(f_afuf_a)\,\theta_{a^{-1}} \\
&= (f_ae_bf_afuf_a)\,\theta_{a^{-1}} \\
&= (e_bf_a \cdot e_bf \cdot uf_a)\,\theta_{a^{-1}} \\
&= (e_bf_a \cdot uf_a)\,\theta_{a^{-1}} \qquad \text{since} \quad e_bf_a \mathscr{D} e_bf \mathscr{D} uf_a \\
&= (f_ae_buf_a)\,\theta_{a^{-1}} \\
&= (f_ae_b)\,\theta_{a^{-1}} \qquad \text{since} \quad e_bu = e_b \\
&= e_{ab}\,.
\end{aligned}$$

Thus we see that $e(fu)\bar{\theta}_{a^{-1}} \mathscr{L} e_{ab}$. Similarly, $(fu)\bar{\theta}_bv \mathscr{R} f_{ab}$. ∎

COROLLARY 2.4. *If*

$$W = W(B, S, \theta) = \{(e, a, f) \in B \times S \times B;\ e \mathscr{L} e_a,\ f \mathscr{R} f_a\}$$

then the prescription

$$(e, a, f)(u, b, v) = (e(fu)\,\bar{\theta}_{a^{-1}},\ ab,\ (fu)\,\bar{\theta}_b v)$$

defines a binary operation on W. ∎

We can in fact say a lot more:

THEOREM 2.5. *$W(B, S, \theta)$ is an orthodox semigroup whose band of idempotents is isomorphic to B.*

Proof. Given (e, a, f), (u, b, v), $(x, c, y) \in W$ the first coordinate of the product $[(e, a, f)(u, b, v)](x, c, y)$ is

$$\begin{aligned}
& e(fu)\,\bar{\theta}_{a^{-1}}[(fu)\,\bar{\theta}_b vx]\,\bar{\theta}_{(ab)^{-1}} \\
&\quad = e(fu)\,\bar{\theta}_{a^{-1}}([(fu)\,\bar{\theta}_{b^{-1}})\,\bar{\theta}_{a^{-1}} && \text{by Lemma 2.2} \\
&\quad = e(f_a fu f_a)\,\theta_{a^{-1}}(f_a[(fu)\,\bar{\theta}_b vx]\,\bar{\theta}_{b^{-1}} f_a)\,\theta_{a^{-1}} \\
&\quad = e(f_a fu f_a[(fu)\,\bar{\theta}_b vx]\,\bar{\theta}_{b^{-1}} f_a)\,\theta_{a^{-1}} \\
&\quad = e(f_a fu f_a[f_b(e_b fu e_b)\,\theta_b vx f_b]\,\theta_{b^{-1}} f_a)\,\theta_{a^{-1}} \\
&\quad = e(f_a fu f_a[f_b(e_b fu e_b)\,\theta_b f_b f_b vx f_b]\,\theta_{b^{-1}} f_a)\,\theta_{a^{-1}} && \text{since } f_b v = v \\
&\quad = e(f_a fu f_a e_b e_b fu e_b e_b(f_b vx f_b)\,\theta_{b^{-1}} f_a)\,\theta_{a^{-1}} && \text{since } f_b\theta_{b^{-1}} = e_b \\
& && \text{and } \theta_b\theta_{b^{-1}} = \theta_b\theta_b^{-1} \\
& && \text{is the identity on its domain} \\
&\quad = e(f_a fu f_a e_b fu(vx)\,\bar{\theta}_{b^{-1}} f_a)\,\theta_{a^{-1}} && \text{since } ue_b = u \\
&\quad = e(f_a fu(vx)\,\bar{\theta}_{b^{-1}} f_a)\,\theta_{a^{-1}} && \text{since } fu \mathscr{D} f_a e_b \\
&\quad = e(fu(vx)\,\bar{\theta}_{b^{-1}})\,\bar{\theta}_{a^{-1}}
\end{aligned}$$

which is the first coordinate of $(e, a, f)[(u, b, v)(x, c, y)]$. Similarly, these products have the same third coordinates. Since their second coordinates are clearly equal, associativity follows.

Now it is clear that $(e, a, f)^2 = (e, a, f)$ implies that $a^2 = a$. Conversely, if $a^2 = a$ then $e_a = aa^{-1} = a = a^{-1}a = f_a$ and so

$$\begin{aligned}
(e, a, f)(e, a, f) &= (e(fe)\bar{\theta}_a,\ a^2,\ (fe)\bar{\theta}_a f) \\
&= (e(f_a fe f_a)\theta_a,\ a,\ (e_a fe e_a)\theta_a f) \\
&= (efe,\ a,\ fef) && \text{since } e_a f = f_a f = f,\ ef_a = ee_a = e \text{ and} \\
& && \theta_a \text{ is the identity on its domain} \\
&= (e, a, f) && \text{since } e \mathscr{L} e_a = f_a \mathscr{R} f \text{ so } e \mathscr{D} f.
\end{aligned}$$

Thus we see that the idempotents of W are precisely those triples of the form (e, a, f) in which a is an idempotent of S. Since the idempotents of S form a subsemigroup, it follows that so also do the idempotents of W.

Given $(e, a, f) \in W$ consider now the triple (f_a, a^{-1}, e_a). Since $e_a = aa^{-1} = (a^{-1})^{-1}a^{-1} = f_{a^{-1}}$ and $f_a = a^{-1}(a^{-1}(a^{-1})^{-1} = e_{a^{-1}}$ we see that $(f_a, a^{-1}, e_a) \in W$. Now

$$\begin{aligned}(e, a, f)(f_a, a^{-1}, e_a) &= (e(ff_a)\,\bar{\theta}_{a^{-1}}, aa^{-1}, (ff_a)\,\bar{\theta}_{a^{-1}}e_a)\\ &= (e(f_aff_a)\,\theta_{a^{-1}}, aa^{-1}, (f_aff_a)\,\theta_{a^{-1}}e_a)\\ &= (e(f_a)\,\theta_{a^{-1}}, aa^{-1}, (f_a)\,\theta_{a^{-1}}e_a)\\ &= (ee_a, aa^{-1}, e_ae_a)\\ &= (e, aa^{-1}, e_a)\end{aligned}$$

and so

$$\begin{aligned}(e, a, f)(f_a, a^{-1}, e_a)(e, a, f) &= (e, aa^{-1}, e_a)(e, a, f)\\ &= (e(e_ae)\,\bar{\theta}_{aa^{-1}}, aa^{-1}a, (e_ae)\,\bar{\theta}_af)\\ &= (e(e_a)\,\bar{\theta}_{aa^{-1}}, a, (e_aee_a)\,\theta_af)\\ &= (ee_a, a, f) \qquad \text{since } \theta_{aa^{-1}} \text{ is the identity on its domain}\\ &= (e, a, f).\end{aligned}$$

Thus we see that W is also regular with (f_a, a^{-1}, e_a) an inverse of (e, a, f).

Let $C = \{(e, a, f) \in W;\ a^2 = a\}$ be the set of idempotents of W and define $\phi\colon C \to B$ by $(e, a, f)\phi = ef$. Then ϕ is a homomorphism; for

$$\begin{aligned}[(e, a, f)(u, b, v)]\phi &= (e(fu)\,\bar{\theta}_{a^{-1}}, ab, (fu)\,\bar{\theta}_bv)\phi\\ &= e(fu)\,\bar{\theta}_{a^{-1}}(fu)\,\bar{\theta}_bv\\ &= ef_afuf_ae_bfue_bv \qquad \text{since } a, b \in E\\ &= efuf_ae_bfuv \qquad \text{since } f_af = f,\ f_b = e_b,\ ue_b = u\\ &= efuv \qquad \text{since } fu\,\mathscr{D}\,f_ae_b\\ &= (e, a, f)\,\phi(u, b, v)\phi.\end{aligned}$$

Now ϕ is injective. For, suppose that $(e, a, f)\phi = (u, b, v)\phi$. Then $ef = uv$. But $e\,\mathscr{L}\,e_a = f_a\,\mathscr{R}\,f$ implies that $ef\,\mathscr{D}\,e_a = a$; and similarly we have $uv\,\mathscr{D}\,e_b = b$. Since e_a, e_b belong to the skeleton E of B and are thus $\mathscr{D}$-equivalent, it follows that $e_a = e_b$ whence $a = b$. Now every element of the $\mathscr{D}$-class of e_a can be written uniquely as xy where $x\,\mathscr{L}\,e_a\,\mathscr{R}\,y$. It follows then that $e = u$ and $f = v$, whence se see that ϕ is injective. Finally, ϕ is surjective. For, given $b \in B$ there exists $e \in E$ such that $e \mathscr{D} b$, whence

$$b = beb = (be, e, eb)\phi.$$

Thus $\phi\colon C \to B$ is an isomorphism. ▮

Remark 2.6. Note that in the case where B is normal the above multiplication simplifies enormously. Indeed, in this case we have

$$(e, e, f)(u, b, v) = (ee_{ab}, ab, f_{ab}v).$$

To see this, observe that

$$\begin{aligned}(fu)\,\bar\theta_{a^{-1}} = (f_a f u f_a)\,\theta_{a^{-1}} &= (f_a f u e_b f_a)\,\theta_{a^{-1}} && \text{since} \quad u = ue_b\\ &= (f_a f e_b u f_a)\,\theta_{a^{-1}} && \text{by normality}\\ &= (f_a f e_b f_a)\,\theta_{a^{-1}} && \text{since} \quad e_b = e_b u\\ &= (f_a e_b f f_a)\,\theta_{a^{-1}} && \text{by normality}\\ &= (f_a e_b f_a)\,\theta_{a^{-1}} && \text{since} \quad f_a = ff_a\\ &= (f_a e_b)\,\theta_{a^{-1}}\\ &= e_{ab}\end{aligned}$$

and similarly that $(fu)\bar\theta_b = f_{ab}$.

This is precisely the multiplication used by Yamada [6], given that we are making a choice of $\mathscr{D}$-transversal.

THEOREM 2.7. *The orthodox semigroup* $W = W(B, S, \theta)$ *is split and* $W/\mathscr{Y} \simeq S$.

Proof. Two elements of W are in the same $\mathscr{Y}$-class if and only if they have the same set of inverses. Now it is clear that if (u, b, v) is an inverse of (e, a, f) then $b = a^{-1}$. On the other hand, consider a product of the form

$$(e, a, f)(u, b, v)(e, a, f).$$

We have seen in the proof of Theorem 2.5 that the first component of this product is

$$\begin{aligned}&e(fu(ve)\,\bar\theta_a)\,\bar\theta_{a^{-1}}\\ &\quad= e(f_a fu(e_a vee_a)\,\theta_a f_a)\,\theta_{a^{-1}}\\ &\quad= e(f_a fu f_a(e_a vee_a)\,\theta_a f_a)\,\theta_{a^{-1}} && \text{since} \quad u = uf_a = ue_a\\ &\quad= e\cdot(f_a fu f_a)\,\theta_{a^{-1}}\cdot e_a vee_a \cdot f_a\theta_{a^{-1}} && \text{since} \quad \theta_a\theta_{a^{-1}} = \theta_{aa^{-1}}\\ &\quad= e\cdot f_a\theta_{a^{-1}}\cdot e_a vee_a\cdot e_a && \text{since} \quad fu \mathscr{D} f_a e_{a^{-1}} = f_a f_a = f_a\\ &\quad= eve && \text{since} \quad ee_a = e\\ &\quad= e && \text{since} \quad v \mathscr{R} f_{a^{-1}} = e_a \mathscr{L} e.\end{aligned}$$

Likewise, the third component of the product is f, whilst the second component is obviously a. We thus see that the set of inverses of (e, a, f) is

$$V(e, a, f) = \{(u, a^{-1}, v);\ u \mathscr{L} e_{a^{-1}} = f_a,\ v \mathscr{R} f_{a^{-1}} = e_a\}.$$

Consequently we see that

$$(e, a, f) \mathscr{Y} (u, b, v) \Leftrightarrow a^{-1} = b^{-1} \Leftrightarrow a = b.$$

Consider now the mapping π: $W/\mathscr{Y} \to W$ given by $\mathscr{Y}_{(e,a,f)}\pi = (e_a, a, f_a)$. That π is a homomorphism follows from the observation that

$$\begin{aligned}
\mathscr{Y}_{(e,a,f)}\pi \cdot \mathscr{Y}_{(u,b,v)}\pi &= (e_a, a, f_a)(e_b, b, f_b) \\
&= (e_a(f_a e_b)\, \bar{\theta}_{a^{-1}},\ ab,\ (f_a e_b)\, \bar{\theta}_b f_b) \\
&= (e_a(f_a e_b f_a)\, \theta_{a^{-1}},\ ab,\ (f_a e_b f_a)\, \theta_b f_b) \\
&= (e_a e_{ab},\ ab,\ f_{ab} f_b) \\
&= (e_{ab},\ ab,\ f_{ab}) \\
&= \mathscr{Y}_{(e,a,f)(u,b,v)}\pi.
\end{aligned}$$

That $\pi\natural = id_{p/\mathscr{Y}}$ now follows from the fact that

$$(e, a, f) \mathscr{Y} (e_a, a, f_a) = \mathscr{Y}_{(e,a,f)}\pi.$$

Thus we see that W is split. Finally, it is clear that

$$W/\mathscr{Y} \simeq \operatorname{Im} \pi \simeq S,$$

the second isomorphism being that given by $(e_a, a, f_a) \leftrightarrow a$. ∎

Our objective now is to show that every split orthodox semigroup is of the form $W(B, S, \theta)$. More precisely:

THEOREM 2.8. *Let T be a split orthodox semigroup with band B of idempotents. If π: $T/\mathscr{Y} \to T$ is a splitting homomorphism then the set $E = B \cap \operatorname{Im} \pi$ of idempotents of* $\operatorname{Im} \pi$ *is a skeleton of B and* $\operatorname{Sp}(E) = \operatorname{Im} \pi$. *Moreover, if θ: $\operatorname{Im} \pi \to T\rho$ is given by $a\theta = \theta_a$, where the domain of θ_a is $aa^{-1}Baa^{-1}$, the codomain of θ_a is $a^{-1}aBa^{-1}a$ and $b\theta_a = a^{-1}ba$, then θ is a triangulation of μ: $\operatorname{Im} \pi \to T_p$ and*

$$T \simeq W(B, \operatorname{Im} \pi, \theta).$$

Proof. $\operatorname{Im} \pi$ is an inverse subsemigroup of T that meets every $\mathscr{Y}$-class exactly once. Consequently, the set $E = B \cap \operatorname{Im} \pi$ of idempotents of $\operatorname{Im} \pi$ meets every $\mathscr{D}$-class of B exactly once and so is a skeleton of B. By Theorem 1.7 it follows that $\operatorname{Im} \pi = \operatorname{Sp}(E)$.

For every $a \in \operatorname{Im} \pi$ let $e_a = aa^{-1}$, $f_a = a^{-1}a$ and define θ_a: $e_aBe_a \to f_aBf_a$ by the prescription $b\theta_a = a^{-1}ba$. Then θ_a is a homomorphism; for if $b = e_abe_a$ and $v = e_ace_a$ then

$$(bc)\theta_a = a^{-1}bca = a^{-1}e_abe_ae_ace_aa = a^{-1}baa^{-1}ca = b\theta_a c\theta_a .$$

In fact θ_a is an isomorphism; for, on the one hand,

$$\begin{aligned} b\theta_a = c\theta_a &\Rightarrow a^{-1}ba = a^{-1}ca \\ &\Rightarrow b = e_abe_a = aa^{-1}baa^{-1} = aa^{-1}caa^{-1} = e_ace_a = c \end{aligned}$$

and, on the other hand, if $x \in f_aBf_a$ then $x = a^{-1}axa^{-1}a = (axa^{-1})\theta_a$ where $axa^{-1} \in e_aBe_a$ since $e_aaxa^{-1}e_a = axa^{-1}$. Moreover, each θ_a preserves the skeleton E; for if $b \in \operatorname{Dom} \theta_a$ is such that $b\theta_a \in E$ then

$$b = e_abe_a = aa^{-1}baa^{-1} = a(b\theta_a)\, a^{-1} = (b\theta_a)\, \mu_{a^{-1}} \in E$$

and if $b \in E$ then $b\theta_a = a^{-1}ba = b\mu_a \in E$. Thus we see that $\theta_a \in T_B$ for every $a \in \operatorname{Im} \pi$.

We note from the semigroup structure that is defined on T_B that if $a, b \in \operatorname{Im} \pi$ then the domain of $\theta_a\theta_b$ is uBu where

$$u = (f_ae_b)\, \theta^{-1} = (a^{-1}abb^{-1})\, \theta_{a^{-1}} = aa^{-1}abb^{-1}a^{-1} = ab(ab)^{-1} = e_{ab}$$

and the codomain of $\theta_a\theta_b$ is vBv where

$$v = (f_ae_b)\theta_b = (e_bf_a)\theta_b = b^{-1}bb^{-1}a^{-1}ab = (ab)^{-1}ab = f_{ab} .$$

Since also, for all x in the domain of $\theta_a\theta_b$, we have

$$x\theta_a\theta_b = (a^{-1}xa)\theta_b = b^{-1}a^{-1}xab = x\theta_{ab}$$

we deduce that the mapping θ: $\operatorname{Im} \pi \to T_B$ described by $a\theta = \theta_a$ is a homomorphism. From the definition of θ_a it is clear that $\hat{\theta}_a = \mu_a$. Consequently, θ is a triangulation of μ: $\operatorname{Im} \pi \to E_E$. We can therefore construct the split orthodox semigroup $W = W(B, \operatorname{Im}, \pi, \theta)$.

Consider now the mapping ψ: $W \to T$ given by

$$(e, a, f)\psi = eaf.$$

Given $(e, a, f), (u, b, v) \in W$ we have

$$
\begin{aligned}
[(e, a, f)(u, b, v)]\psi &= (e(fu)\,\bar{\theta}_{a^{-1}},\, ab,\, (fu)\,\bar{\theta}_b v)\psi \\
&= e(fu)\,\bar{\theta}_{a^{-1}} \cdot ab \cdot (fu)\,\bar{\theta}_b v \\
&= e(f_a fu f_a)\,\theta_{a^{-1}} ab(e_b fu e_b)\,\theta_b v \\
&= eaf_a fu f_a a^{-1}abb^{-1}e_b fu e_b bv \\
&= eafua^{-1}abb^{-1}fubv \\
&= eafubb^{-1}a^{-1}afubv \\
&= eafufubv \qquad \text{since} \quad u\,\mathscr{L}\,e_b = bb^{-1}, f\,\mathscr{R}\,f_a = a^{-1}a \\
&= eafubv \qquad \text{since} \quad fufu = fu \\
&= (e, a, f)\,\psi(u, b, v)\psi.
\end{aligned}
$$

Thus ψ is a homomorphism.

To show that ψ is injective, we note first that

$$
\begin{aligned}
eaf \cdot a^{-1} \cdot eaf &= eafa^{-1}a \cdot a^{-1} \cdot aa^{-1}eaf \\
&= eaa^{-1}a \cdot a^{-1} \cdot aa^{-1}af \qquad \text{since } f\,\mathscr{R}\,f_a = a^{-1}a,\ e\,\mathscr{L}\,e_a = aa^{-1} \\
&= eaf
\end{aligned}
$$

and similarly that

$$a^{-1} \cdot eaf \cdot a^{-1} = a^{-1} \cdot aa^{-1}eafa^{-1}a \cdot a^{-1} = a^{-1}aa^{-1} = a^{-1}.$$

Consequently we have $\mathscr{Y}_{eaf}\pi = a$. Suppose then that $(e, a, f)\psi = (u, b, v)\psi$. Then $eaf = ubv$ and so $a = \mathscr{Y}_{eaf}\pi = \mathscr{Y}_{nby}\pi = b$ whence

$$f = f_a f = a^{-1}af = a^{-1}aa^{-1}eaf = b^{-1}bb^{-1}ubv = b^{-1}bv = v$$

and similarly $e = u$. Thus we see that ψ is injective.

Finally, given $x \in T$ let $\tilde{x} = \mathscr{Y}_x\pi$. Since $\mathscr{Y}_x\pi \in \mathscr{Y}_x$ it is clear that $\tilde{x}^{-1}$ is an inverse of x in T. Consequently

$$x\tilde{x}^{-1}\tilde{x}\tilde{x}^{-1} = x\tilde{x}^{-1} \qquad \text{and} \qquad \tilde{x}\tilde{x}^{-1}x\tilde{x}^{-1} = \tilde{x}\tilde{x}^{-1}$$

whence $x\tilde{x}^{-1}\,\mathscr{L}\,e_{\tilde{x}} = \tilde{x}\tilde{x}^{-1}$. Similarly, we have $\tilde{x}^{-1}x\,\mathscr{R}\,f_{\tilde{x}} = \tilde{x}^{-1}\tilde{x}$. Since now

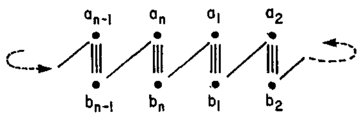
$$x = x\tilde{x}^{-1}x = x\tilde{x}^{-1} \cdot \tilde{x} \cdot \tilde{x}^{-1}x$$

it follows that $(x\tilde{x}^{-1}, \tilde{x}, \tilde{x}^{-1}x) \in W$ with $(x\tilde{x}^{-1}, \tilde{x}, \tilde{x}^{-1}x)\psi = x$. Thus ψ is surjective and so is an isomorphism. ∎

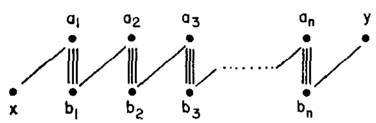
3. Interpolation and Link Properties

Our objective now is to determine analogues of the previous results for the case of *ordered* orthodox semigroups. In order to define the notion of splitting in this case, we require the homomorphisms in question to be isotone; and for this it is essential to be able to define an ordering on $T/\mathscr{Y}$ that will make $T/\mathscr{Y}$ into an ordered semigroup. In this section we prepare the way for this by considering certain orderings on quotient sets.

We recall some terminology and results from [1]. An equivalence relation ρ on an ordered set E is said to be *regular* if there is an ordered set F and an isotone mapping $f: E \to F$ such that ρ is the equivalence relation associated with f in that $x\rho y \Leftrightarrow f(x) = f(y)$. Equivalently, ρ is regular on E if E/ρ can be ordered in such a way that the natural map $\natural: E \to E/\rho$ is isotone. A *closed bracelet* modulo ρ is a diagram of the form



in which three vertical lines denote equivalence modulo ρ and the single lines are as in a Hasse diagram. An *open bracelet* modulo ρ is a diagram of the form



In this, x is called the *initial clasp* and y the *terminal clasp*. By Theorem 6.1 of [1], ρ is regular on E if and only if, in every closed bracelet modulo ρ, all the elements of the bracelet belong to the same class modulo ρ. In this case E/ρ can be ordered in a natural way by

$$\rho_x \leqslant \rho_y \Leftrightarrow \begin{cases}\text{there is an open bracelet modulo } \rho \text{ with} \\ \text{initial clasp } x \text{ and terminal clasp } y.\end{cases}$$

By Theorem 6.2 of [1] regular equivalences have convex classes. A regular equivalence ρ on an ordered set E is *strongly upper regular* if it satisfies the *link property*; this property is pictorially illustrated by

By Theorem 6.6 of [1], ρ is strongly upper regular if and only if it has convex classes and satisfies the link property. In this case, by applying the link property repeatedly to any open bracelet we see that the above ordering on E/ρ reduces to

$$\rho_x \leqslant \rho_y \Leftrightarrow (\forall x^* \in \rho_x)(\exists y^* \in \rho_y) \qquad x^* \leqslant y^*.$$

In relation to the above, we now introduce the following more general concept.

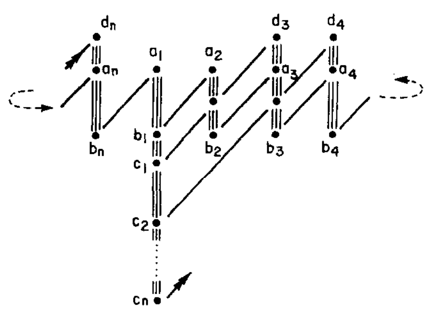
DEFINITION 3.1. We shall say that an equivalence relation ρ on an ordered set E has the *interpolation property* if

It is clear from the above definitions that if ρ satisfies either the link property or its dual then ρ satisfies the interpolation property (in the link property take $x = a, y = b$ and in its dual take $y = c, z = d$).

DEFINITION 3.2. By an *inter-regular* equivalence on E we mean a regular equivalence ρ on E that satisfies the interpolation property.

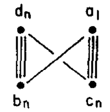
THEOREM 3.3. *An equivalence relation ρ on an ordered set E is inter-regular if and only if ρ has convex classes and satisfies the interpolation property.*

Proof. The necessity is clear. As for sufficiency, we apply the interpolation property repeatedly to a closed bracelet modulo ρ, thereby obtaining a diagram of the form

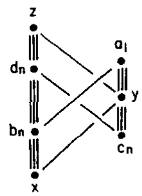


More precisely, begin with $b_1 \leqslant a_2 \equiv b_2 \leqslant a_3$ and apply the interpolation property to obtain c_1, d_3; now consider $c_1 \leqslant d_3 \equiv b_3 \leqslant a_4$ to obtain c_2, d_4; etc.

This process is carried out $n - 2$ times to obtain the above diagram. Now from this diagram we extract the diagram



Applying the interpolation property yet again (since $b_n \leqslant a_1 \equiv c_n \leqslant d_n$) we obtain a diagram



The convexity of the ρ-classes now shows that the above two classes coalesce, so that $b_n \equiv a_1$. Applying this argument repeatedly to the original closed bracelet, we see that all the elements of the bracelet are ρ-equivalent, whence the result follows. ∎

In the case where ρ is inter-regular we can apply the interpolation property to any open bracelet and thereby see that the natural ordering on E/ρ reduces to

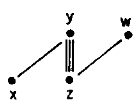
$$\rho_x \leqslant \rho_y \Leftrightarrow (\exists x^* \in \rho_x)(\exists y^* \in \rho_y) \qquad x^* \leqslant y^*.$$

DEFINITION 3.4. If X, Y are ordered sets then an isotone map $\theta: X \to Y$ is said to *split* if there is an isotone map $\pi: Y \to X$ such that $\pi\theta = id_Y$.

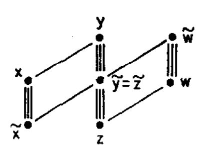
Our interest in inter-regular equivalences is now justified by the following result.

THEOREM 3.5. *Let X and Y be ordered sets and let $\theta: X \to Y$ be an isotone map that splits. Then the equivalence relation ρ associated with θ is inter-regular and* Im θ *is order-isomorphic to X/ρ.*

Proof. Since θ is isotone it is immediate that the ρ-classes are convex. If $\pi: Y \to X$ is an isotone map such that $\pi\theta = id_Y$ then $\theta\pi\theta = \theta$ then $\theta\pi\theta = \theta$ shows that, for every $x \in X$, $x\theta\pi$ and x are in the same ρ-class. Writing $x\theta\pi$ as $\tilde{x}$, it is clear that, modulo ρ,



implies



and consequently ρ is inter-regular.

Now let $\natural: X \to X/\rho$ be the canonical map and let $\theta^+: X \to \operatorname{Im}\theta$ be the map induced by θ, in that $(\forall x \in X) x\theta^+ = x\theta$. If $\pi^+: \operatorname{Im}\theta \to X$ is the restriction of π to $\operatorname{Im}\theta$ then clearly $\pi^+\theta^+ = id_{\operatorname{Im}\theta}$. Now there is a unique bijection $b: X/\rho \to \operatorname{Im}\theta$ such that $\natural b = \theta^+$ (namely, that given by $\rho_x \mapsto \theta(x)$) so $\pi^+ \natural\, b = \pi^+\theta^+ = id_{\operatorname{Im}\theta}$ shows that $\pi^+\natural = b^{-1}$ is a bijection from $\operatorname{Im}\theta$ onto X/ρ. Since π^+ and $\natural$ are isotone, this bijection is isotone. It suffices, therefore, to show that b is also isotone. Suppose then that $\rho_x \leqslant \rho_y$. Then there exist $x^* \in \rho_x$ and $y^* \in \rho_y$ such that $x^* \leqslant y^*$. Since θ^+ is isotone we then have

$$\rho_x b = x \natural\, b = x^* \natural\, b = x^*\theta^+ \leqslant y^*\theta^+ = y^* \natural\, b = y \natural\, b = \rho_y b$$

and so b is indeed isotone. ∎

It follows from the above result that, when we come to consider the notion of split ordered orthodox semigroup in the next section, the congruence $\mathscr{Y}$ will necessarily be inter-regular. But before proceeding to this case, we pause to consider situations concerning ordered orthodox semigroups T in which properties of $\mathscr{Y}$ are determined by properties of Green's relations on the band B of idempotents of T. We first consider the following notion which will play a prominent role in the next section.

DEFINITION 3.6. Let D be an ordered regular semigroup. Then we shall say that the ordering of D is *amenable* if

$$x \leqslant y \Rightarrow (\exists x', x'' \in V(x))(\exists y', y'' \in V(y))\; xx' \leqslant yy',\; x''x \leqslant y''y.$$

Note that if S is an ordered inverse semigroup then, since inverses are unique, the ordering of S is amenable if and only if

$$x \leqslant y \Rightarrow xx^{-1} \leqslant yy^{-1},\; x^{-1}x \leqslant y^{-1}y$$

which is equivalent to saying that Green's relations $\mathscr{R}$, $\mathscr{L}$ are regular. In particular, the natural order on an inverse semigroup is clearly amenable. Note also that every ordered band is amenably ordered (take $x'' = x' = x$ and $y'' = y' = y$).

THEOREM 3.7. *Let T be an amenably ordered orthodox semigroup. Then $\mathscr{Y}$ satisfies the interpolation property on T if and only if $\mathscr{D}$ satisfies the interpolation property on the band B of idempotents of T.*

Proof. Suppose first that $\mathscr{D}$ has the interpolation property on B and let $a \leqslant b \mathrel{\mathscr{Y}} c \leqslant d$ in T. Since T is amenably ordered there exists $a' \in V(a)$, $b' \in V(b)$, $c' \in V(c)$, $d' \in V(d)$ such that $aa' \leqslant bb'$ and $cc' \leqslant dd'$. Since $bb' \mathrel{\mathscr{Y}} cb' \mathrel{\mathscr{Y}} cc'$ we have, in B, $bb' \mathrel{\mathscr{D}} cc'$. By the interpolation property on B there exist $e_1 \leqslant e_2 \leqslant e_3$ in B such that

$$e_1 \mathrel{\mathscr{D}} aa', \qquad e_2 \mathrel{\mathscr{D}} bb', \qquad e_3 \mathrel{\mathscr{D}} dd'.$$

Similarly, there exist $f_1 \leqslant f_2 \leqslant f_3$ in B such that

$$f_1 \mathscr{D} a''a, \qquad f_2 \mathscr{D} b''b, \qquad f_3 \mathscr{D} d''d.$$

Then we have

$$e_1 a f_1 \mathscr{Y} aa'aa''a = a;$$
$$e_2 b f_2 \mathscr{Y} b \mathscr{Y} c \mathscr{Y} e_2 c f_2;$$
$$e_3 d f_3 \mathscr{Y} d;$$
$$e_1 a f_1 \leqslant e_2 b f_2;$$
$$e_2 c f_2 \leqslant e_3 d f_3 .$$

To show that $\mathscr{Y}$ has the interpolation property on T, it therefore suffices to show that $e_2bf_2 = e_2cf_2$. Now

$$\begin{aligned} e_2bf_2 \cdot f_2b'e_2 &= e_2b \cdot b'bf_2b'b \cdot b'e_2 \\ &= e_2b \cdot b'b \cdot b'e_2 \qquad \text{since } f_2 \mathscr{D} b'b \\ &= e_2bb'e_2 \\ &= e_2 \qquad\qquad\qquad \text{since } e_2 \mathscr{D} bb'. \end{aligned}$$

Thus we have $e_2bf_2 \mathscr{R} e_2$ abd similarly $e_2cf_2 \mathscr{R} e_2$. Likewise we can show that $e_2bf_2 \mathscr{L} f_2 \mathscr{L} e_2cf_2$. Thus $(e_2bf_2, e_2cf_2) \in \mathscr{Y} \cap \mathscr{H}$ and so $e_2bf_2 = e_2cf_2$ as required.

Conversely, suppose that $\mathscr{Y}$ has the interpolation property on T and let $a \leqslant b \mathscr{Y} c \leqslant d$ with $a, b, c, d \in B$. Then there exist $x \leqslant y \leqslant z$ in T such that $a \mathscr{Y} x$, $b \mathscr{Y} y$, $d \mathscr{Y} z$. Since $\mathscr{Y}$ is idempotent-determined, it follows that $x, y, z \in B$. Hence $\mathscr{D}$ has the interpolation property on B. ∎

THEOREM 3.8. *Let T be an amenably ordered orthodox semigroup. Then $\mathscr{Y}$ satisfies the (dual) link property on T if and only if $\mathscr{L}$ and $\mathscr{R}$ satisfy the (dual) link property on the band B of idempotents of T.*

Proof. We establish the result for the link property; that for the dual link property follows by replacing $\leqslant$ at every stage by $\geqslant$.

Suppose first that $\mathscr{L}$ and $\mathscr{R}$ satisfy the link property on B and let $x \leqslant y$ in T. Then there exist $x', x'' \in V(x)$ and $y', y'' \in V(y)$ such that $xx' \leqslant yy'$, $x''x \leqslant y''y$. We deduce from $xx'' \mathscr{R} xx' \leqslant yy'$ and the link property on B that there exists $y''' \in V(y)$ such that $xx'' \leqslant yy'''$. Now $yy''' \mathscr{R} y \mathscr{L} y''y$ implies that there exists $z \in V(y)$ such that $yz = yy'''$ and $zy = y''y$. Thus we see that the link property on B yields

$$x \leqslant y \Rightarrow (\exists x'' \in V(x))(\exists z \in V(y))xx'' \leqslant yz,\ x''x \leqslant zy.$$

[Put another way, in the definition of amenable order we can take $x'' = x'$ and $y'' = y'$.] To show now that $\mathscr{Y}$ satisfies the link property on T, suppose that $c\,\mathscr{Y}\,b \leqslant a$. Then, by the above observation, there exist $a' \in V(a)$, $b' \in V(b) = V(c)$ such that $bb' \leqslant aa'$ and $b'b \leqslant a'a$. Thus, noting that $c\,\mathscr{Y}\,b$ implies $cb', b'c \in B$, we deduce on the one hand from $cb'\,\mathscr{L}\,bb' \leqslant aa'$ that there exists $v \in B$ such that $cb' \leqslant v$ with $v\,\mathscr{L}\,aa'\,\mathscr{L}\,a'$; and on the other hand from $b'c\,\mathscr{R}\,b'b \leqslant a'a$ that there exists $w \in B$ such that $b'c \leqslant w$ with $w\,\mathscr{R}\,a'a\,\mathscr{R}\,a'$. Now since $v\,\mathscr{L}\,a'\,\mathscr{R}\,w$ there is an inverse x of a' with $a'x = w$ and $xa' = v$. Consequently

$$c = cb'bb'c \leqslant vbw \leqslant vaw = xa'aa'x = xa'x = x$$

and, since $a' \in V(x) \cap V(a)$, $a\,\mathscr{Y}\,x$. Thus we see that $\mathscr{Y}$ satisfies the link property on T.

Conversely, suppose that $\mathscr{Y}$ satisfies the link property on T and let $c\,\mathscr{L}\,b \leqslant a$ in B. Then for some $x \in T$ we have $c \leqslant x\,\mathscr{Y}\,a$. Since $\mathscr{Y}$ is idempotent-determined, we have $x \in B$. Furthermore, $c\,\mathscr{L}\,b$ implies $cb = c$. Now let $d = xa$; then $d = xa \geqslant ca \geqslant cb = c$ while $da = a$ and $ad = axa = a$, so that $d\,\mathscr{L}\,a$. In summary, therefore, we have

$$c\,\mathscr{L}\,b \leqslant a \Rightarrow (\exists d \in B) \qquad c \leqslant d\,\mathscr{L}\,a.$$

Thus $\mathscr{L}$ satisfies the link property on B; and similarly so does $\mathscr{R}$. ∎

COROLLARY 3.9. *Let B be an ordered band. Then $\mathscr{D}$ satisfies the (dual) link property if and only if $\mathscr{L}$ and $\mathscr{R}$ satisfy the (dual) link property.* ∎

THEOREM 3.10. *Let T be an ordered orthodox semigroup. Then Y has convex classes on T if and only if $\mathscr{D}$ has convex classes on the band B of idempotents of T.*

Proof. Necessity is obvious. As for sufficiency, suppose that $\mathscr{D}$ has convex classes on B and that $a \leqslant b \leqslant c$ in T with $a\,\mathscr{Y}\,c$. Given $a' \in V(a) = V(c)$ we have $a' = a'aa' \leqslant a'ba' \leqslant a'ca' = a'$ and so $a' = a'ba'$. Consequently $a'b$ and ba' are idempotents. Now, on passing to quotients modulo $\mathscr{Y}$ from $a' = a'ba'$, and taking inverses in $T/\mathscr{Y}$, we see that there exists $b' \in V(b)$ such that $a\,\mathscr{Y}\,ab'a$ from which we obtain

$$aa'\,\mathscr{Y}\,ab'aa'\,\mathscr{Y}\,aa'ab' = ab'.$$

But $ab' \leqslant bb' \leqslant cb'$ with ab', cb' idempotents that are $\mathscr{D}$-equivalent (since $\mathscr{Y}$ is idempotent-determined and $a\,\mathscr{Y}\,c$). Thus, by the convexity of the $\mathscr{D}$-classes on B, we have $bb'\,\mathscr{D}\,ab'$ and similarly $b'b\,\mathscr{D}\,b'a$. Consequently

$$a = aa'a\,\mathscr{Y}\,ab'a\,\mathscr{Y}\,bb'a\,\mathscr{Y}\,bb'b = b$$

and so the $\mathscr{Y}$-classes are convex on T. ∎

COROLLARY 3.11. *Let T be an amenably ordered orthodox semigroup. Then $\mathscr{Y}$ is inter-regular on T if and only if $\mathscr{D}$ is inter-regular on B.*

Proof. Immediate from Theorems 3.3 and 3.7. ∎

COROLLARY 3.12. *Let T be an amenably ordered orthodox semigroup. Then $\mathscr{Y}$ is strongly upper (lower) regular on T if and only if $\mathscr{L}$ and $\mathscr{R}$ are strongly upper (lower) regular on B.*

Proof. Immediate from Theorem 3.8. ∎

COROLLARY 3.13. *Let B be an ordered band. Then D is strongly upper (lower) regular if and only if $\mathscr{L}$ and $\mathscr{R}$ are strongly upper (lower) regular.*

Proof. Immediate from Corollary 3.9. ∎

In the particular case where the idempotents satisfy the condition $e \leqslant f \Rightarrow e = efe$, a situation that we shall encounter later, we have the following result.

THEOREM 3.14. *Let T be an amenably ordered orthodox semigroup. Suppose that $e^2 = e \leqslant f = f^2 \Rightarrow e = efe$. Then $\mathscr{Y}$ is strongly lower regular. Moreover, $T/\mathscr{Y}$ is amenably ordered.*

Proof. On passing to quotients modulo $\mathscr{Y}$ we see that, relative to the natural order $\preccurlyeq$ on T/Y,

$$e^2 = e \leqslant f = f^2 \Rightarrow e/\mathscr{D} \preccurlyeq f/\mathscr{D}.$$

It follows immediately that the $\mathscr{D}$-classes are convex on the band B of idempotents of T. Now $\mathscr{D}$ is also strongly lower regular on B. For, suppose that $e \leqslant f \mathscr{D} a$ with $e, f, a \in B$; then $aea \leqslant afa = a$ and since $e \leqslant f$ we have $e = efe \mathscr{D} fef \mathscr{D} aea \leqslant a$. It now follows by Corollaries 3.12 and 3.13 that $\mathscr{Y}$ is strongly lower regular.

To show that $T/\mathscr{Y}$ is amenably ordered (relative to the ordering immediately preceding Definition 3.1) it suffices to show that

$$\mathscr{Y}_a \leqslant \mathscr{Y}_b \Rightarrow (\forall a' \in V(a))(\forall b' \in V(b)) \begin{cases} \mathscr{Y}_{aa'} = \mathscr{Y}_a \mathscr{Y}_a^{-1} \leqslant \mathscr{Y}_b \mathscr{Y}_b^{-1} = \mathscr{Y}_{bb'}\,; \\ \mathscr{Y}_{a'a} = \mathscr{Y}_a^{-1} \mathscr{Y}_a \leqslant \mathscr{Y}_b^{-1} \mathscr{Y}_b = \mathscr{Y}_{b'b}\,. \end{cases}$$

Now if $\mathscr{Y}_a \leqslant \mathscr{Y}_b$ then for all $a' \in V(a)$ there exists $b^* \in V(b)$ such that $a' \leqslant b^*$. Since T is amenably ordered there then exist $a'', a''' \in V(a')$ and $b'', b''' \in V(b^*)$ such that $a'a'' \leqslant b^*b''$ and $a'''a' \leqslant b'''b^*$. In fact, from the proof of Theorem 3.8, we can take $a''' = a''$ and $b''' = b''$. Since $\natural\colon T \to T/\mathscr{Y}$ is isotone and $a'' \mathscr{Y} a$, $b'' \mathscr{Y} b$ we deduce that $\mathscr{Y}_{a'a} \leqslant \mathscr{Y}_{b^*b}$ and $\mathscr{Y}_{aa'} \leqslant \mathscr{Y}_{bb^*}$. The result now follows from the fact that $\mathscr{Y}_{b^*} = \mathscr{Y}_{b'}$ for all $b' \in V(b)$. ∎

4. Split Ordered Orthodox Semigroups

DEFINITION 4.1. Let $B = \bigcup\{B_\alpha; \alpha \in Y\}$ be an ordered band. Then we say that B is *split* if

(1) $B/\mathscr{D}$ can be ordered in such a way that $\natural: B \to B/\mathscr{D}$ is isotone;

(2) there is an isotone homomorphism $\pi: B/\mathscr{D} \to B$ such that $\pi\natural = id_{B/\mathscr{D}}$.

Note that if B is split then, by Theorem 3.5, $\mathscr{D}$ is inter-regular.

DEFINITION 4.2. Let $B = \bigcup\{B_\alpha; \alpha \in Y\}$ be an ordered band. Then by a *skeleton* of B we mean a subset $E = \{x_\alpha; \alpha \in Y\}$ of B such that

(1) $x_\alpha \in B_\alpha$ for every $\alpha \in Y$;

(2) $x_\alpha x_\beta = x_{\alpha\beta} = x_\beta x_\alpha$ for all $\alpha, \beta \in Y$;

(3) if $x \in B_\alpha$, $y \in B_\beta$ are such that $x \leqslant y$ then $x_\alpha \leqslant x_\beta$.

Note that in an ordered almost commutative band any $\mathscr{D}$-transversal satisfies (1), (2) of Definition 4.2; but there may be no $\mathscr{D}$-transversal satisfying (3), for the $\mathscr{D}$-classes need not be convex.

LEMMA 4.3. *An ordered band B is split if and only if it has a skeleton. If $\pi: B/\mathscr{D} \to B$ is an isotone splitting homomorphism then* Im π *is a skeleton of B.*

Proof. Recall the proof of Lemma 1.3. If B is split then $\natural$, π are isotone, whence $x \leqslant y$ with $x \in B_\alpha$, $y \in B_\beta$ implies $x_\alpha = B_\alpha\pi \leqslant B_\beta\pi = x_\beta$.

Conversely, if $E = \{x_\alpha; \alpha \in Y\}$ is a skeleton of B define $\leqslant$ on $B/\mathscr{D}$ by $B_\alpha \leqslant B_\beta \Leftrightarrow x_\alpha \leqslant x_\beta$. This is clearly an ordering on $B/\mathscr{D}$ and (3) of the above definition shows that $\natural: B \to B/\mathscr{D}$ is isotone. Defining $\pi: B/\mathscr{D} \to B$ as in Lemma 1.3, namely by $B_\alpha\pi = x_\alpha$, we see that π is an isotone splitting homomorphism. ▮

If now B is an ordered band with skeleton E we shall extend our notation to denote by T_B the set of all skeleton-preserving ordered band isomorphisms between ordered sub-bands of the form eBe where $e \in E$.

LEMMA 4.4. *Given $\theta, \phi \in T_B$ define*

$$\theta \leqslant \phi \Leftrightarrow (\forall b \in B) \qquad b\bar{\theta} \leqslant b\bar{\phi}.$$

Then $(T_B, \leqslant)$ is an ordered inverse semigroup.

Proof. It is clear that $\leqslant$ is both reflexive and transitive on T_B. Suppose now that $\theta \leqslant \phi$ and $\phi \leqslant \theta$. Then clearly $\bar{\theta} = \bar{\phi}$ and so

$$f_\theta = e_\theta\theta = e_\theta\bar{\theta} = e_\theta\bar{\phi} = (e_\phi e_\theta e_\phi)\phi \in f_\phi B f_\phi$$

whence $f_\theta = f_\phi f_\theta f_\phi = f_\phi f_\theta$. Similarly $f_\phi = f_\theta f_\phi$ and so $f_\theta = f_\phi$. Since then $e_\phi \phi = f_\phi = f_\theta = e_\theta \theta = (e_\phi e_\theta e_\phi)\phi$ we deduce that $e_\phi = e_\phi e_\theta e_\phi = e_\phi e_\theta$; and similarly $e_\theta = e_\theta e_\phi$ so that $e_\theta = e_\phi$. Consequently Dom θ = Dom ϕ and Cod θ = Cod ϕ. That $\theta = \phi$ now follows from the fact that θ is induced by $\bar{\theta}$. Thus $\leqslant$ is an ordering on T_B. That the inverse semigroup T_B is an ordered semigroup now follows from the fact that if $\theta_1, \theta_2, \phi_1, \phi_2 \in T_B$ are such that $\theta_1 \leqslant \theta_2$ and $\phi_1 \leqslant \phi_2$ then, by Lemma 2.2,

$$(\forall b \in B) \qquad b\overline{\theta_1\phi_1} = b\bar{\theta}_1\bar{\phi}_1 \leqslant b\bar{\theta}_2\bar{\phi}_1 \leqslant b\bar{\theta}_2\bar{\phi}_2 = b\overline{\theta_2\phi_2}$$

and so $\theta_1\phi_1 \leqslant \theta_2\phi_2$. ∎

Suppose now that S is an amenably ordered inverse semigroup. If E is the semilattice of idempotents of S and $\mu: S \to T_E$ is the Munn homomorphism then for all $a, b \in S$ we have

$$a \leqslant b \Rightarrow \mu_a \leqslant \mu_b, \quad \mu_{a^{-1}} \leqslant \mu_{b^{-1}}.$$

In fact, if $a \leqslant b$ and $x \in E$ then $xa \leqslant xb$ and so

$$x\mu_a = a^{-1}xa = (xa)^{-1}xa \leqslant (xb)^{-1}xb = b^{-1}xb = x\mu_b;$$

and $ax \leqslant bx$ so that

$$x\mu_{a^{-1}} = axa^{-1} = ax(ax)^{-1} \leqslant bx(bx)^{-1} = bxb^{-1} = x\mu_{b^{-1}}.$$

This leads to the following notion.

DEFINITION 4.5. Let B be an ordered band with skeleton E and let

$$L(E) = \{x \in B; (\exists e \in E)\, x \mathscr{L} e\};$$

$$R(E) = \{x \in B; (\exists e \in E)\, x \mathscr{R} e\}.$$

If S is an ordered inverse semigroup with semilattice of idempotents E and if $\theta: S \to T_B$ is a triangulation of $\mu: S \to T_E$ then we shall say that θ is *balanced if*

$$a \leqslant b \Rightarrow \begin{cases} (\forall x \in L(E)) & x\bar{\theta}_{a^{-1}} \leqslant x\bar{\theta}_{b^{-1}}; \\ (\forall x \in R(E)) & y\bar{\theta}_a \leqslant y\bar{\theta}_b. \end{cases}$$

THEOREM 4.6. *Let B be an ordered band and let S be an inverse semigroup. Suppose that the semilattice E of idempotents of S is a skeleton of B and let $\theta: S \to T_B$ be a triangulation of $\mu: S \to T_E$. Suppose further that S is amenably ordered. Then the following statements are equivalent*:

(1) $W = W(B, S, \theta)$ *is an ordered semigroup under the cartesian ordering*;

(2) θ *is balanced.*

Proof. (1) $\Rightarrow$ (2). Suppose that (1) holds. Given $a, b \in S$ with $a \leqslant b$ and given $x \in L(E)$ with, say, $x \mathscr{L} e_c$ then, since S is amenably ordered, we have $e_a = aa^{-1} \leqslant bb^{-1} = e_b$ and $f_a = a^{-1}a \leqslant b^{-1}b = f_b$ so that $(e_a, a, f_a) \leqslant (e_b, b, f_b)$ and consequently

$$(e_a, a, f_a)(x, c, f_c) \leqslant (e_b, b, f_b)(x, c, f_c)$$

whence we deduce that $e_a(f_a x)\bar{\theta}_{a^{-1}} \leqslant e_b(f_b x)\bar{\theta}_{b^{-1}}$. But

$$e_a(f_a x)\,\bar{\theta}_{a^{-1}} = f_a\bar{\theta}_{a^{-1}}(f_a x f_a)\,\theta_{a^{-1}} = (f_a x f_a)\,\theta_{a^{-1}} = x\bar{\theta}_{a^{-1}}.$$

Thus we see that $x\bar{\theta}_{a^{-1}} \leqslant x\bar{\theta}_{b^{-1}}$. Similarly we can show that if $y \in R(E)$ then $y\bar{\theta}_a \leqslant y\bar{\theta}_b$. Thus θ is balanced.

(2) $\Rightarrow$ (1): Suppose now that θ is balanced and let (e, a, f), (u, b, v), (x, c, y) be elements of W with $(e, a, f) \leqslant (u, b, v)$. Then

$$(e, a, f)(x, c, y) = (e(fx)\,\bar{\theta}_{a^{-1}}, ac, (fx)\,\bar{\theta}_c y);$$
$$(u, b, v)(x, c, y) = (u(vx)\,\bar{\theta}_{b^{-1}}, bc, (vx)\,\bar{\theta}_c y).$$

Now since $f \leqslant v$ we have $fx \leqslant vx$; and S is amenably ordered, so that from $a \leqslant b$ we obtain $e_a \leqslant e_b$ and $f_a \leqslant f_b$. Thus

$$f_a f x f_a \leqslant f_b v x f_b. \tag{*}$$

We now observe that $f_a f x f_a \in L(E)$. For $fx \mathscr{D} f_a x \mathscr{D} f_a e_c$ in B and so

$$\begin{aligned} f_a e_c \cdot f_a f x f_a &= f_a e_c f_a f x e_c f_a && \text{since } x = xe_c \\ &= f_a e_c f_a && \text{since } fx \mathscr{D} f_a e_c \\ &= f_a e_c, \\ f_a f x f_a \cdot f_a e_c &= f_a f x e_c f_a \\ &= f_a f x f_a && \text{since } x = xe_c. \end{aligned}$$

Thus we see that $f_a f_x f_a \mathscr{L} f_a e_c \in E$.

Consequently, since θ is balanced,

$$\begin{aligned} (fx)\,\bar{\theta}_{a^{-1}} = (f_a f x f_a)\,\theta_{a^{-1}} &\leqslant (f_a f x f_a)\,\theta_{b^{-1}} \\ &\leqslant (f_b v x f_b)\,\theta_{b^{-1}} \qquad \text{by } (*) \\ &= (vx)\,\bar{\theta}_{b^{-1}}. \end{aligned}$$

Hence $e(fx)\bar{\theta}_{a^{-1}} \leqslant u(vx)\bar{\theta}_{b^{-1}}$. A similar argument can be applied to the third coordinates. We then obtain $(e, a, f)(x, c, y) \leqslant (u, b, v)(x, c, y)$. Dually,

$(x, c, y)(e, a, f) \leqslant (x, c, y)(u, b, v)$ and we conclude that W is an ordered semigroup under the cartesian order. ∎

We can now consider the notion of splitting in the case of ordered orthodox semigroups.

DEFINITION 4.7. Let T be an ordered orthodox semigroup. Then we shall say that T is *split* if

(1) $T/\mathscr{Y}$ can be ordered such that $\natural: T \to T/\mathscr{Y}$ is isotone;

(2) there is an isotone homomorphism $\pi: T/\mathscr{Y} \to T$ such that $\pi\natural = id_{E/\mathscr{D}}$.

Note that if T is split then, by Theorem 3.5, $\mathscr{Y}$ is inter-regular.

Remark 4.8. Note that in a split ordered orthodox semigroup $\natural$ and π are always isotone; in an ordered split orthodox semigroup they need not be.

Continuing the notation of Theorem 4.6, we now have:

THEOREM 4.9. *If S is amenably ordered and θ is balanced then $W = W(B, S, \theta)$ is a split amenably ordered orthodox semigroup and the ordered semigroups $W/\mathscr{Y}$ and S are isomorphic. Moreover, the band of idempotents of W is isomorphic to the ordered band B.*

Proof. By Theorem 4.6, W is an ordered semigroup under the cartesian order. To show that W is amenably ordered consider $(e, a, f), (u, b, v) \in W$ with $(e, a, f) \leqslant (u, b, v)$. Since $(f_a, a^{-1}, e_a) \in V(e, a, f)$ with

$$(e, a, f)(f_a, a^{-1}, e_a) = (e, aa^{-1}, aa^{-1})$$

(see the proof of Theorem 2.5) and since $e \leqslant u$, $a \leqslant b$ it follows from the fact that S is amenably ordered that

$$(e, a, f)(f_a, a^{-1}, e_a) \leqslant (u, b, v)(f_b, b^{-1}, e_b).$$

Similarly we have

$$(f_a, a^{-1}, e_a)(e, a, f) \leqslant (f_b, b^{-1}, e_b)(u, b, v)$$

and so W is amenably ordered.

To show that W is split, define the relation $\leqslant$ on $W/\mathscr{Y}$ by

$$\mathscr{Y}_{(e,a,f)} \leqslant \mathscr{Y}_{(u,b,v)} \Leftrightarrow a \leqslant b.$$

Since (from the proof of Theorem 2.7) we have

$$(e, a, f) \,\mathscr{Y}\, (u, b, v) \Leftrightarrow a = b,$$

it is clear that $\leqslant$ is an ordering on $W/\mathscr{Y}$ and that $\natural: W \to W/\mathscr{Y}$ is an isotone homomorphism. Now define $\pi: W/\mathscr{Y} \to W$ by

$$\mathscr{Y}_{(e,a,c)}\pi = (e_a, a, f_a).$$

From the proof of Theorem 2.7, π is a homomorphism; and since S is amenably ordered π is clearly isotone. Since $\pi\natural$ is the identity map on $W/\mathscr{Y}$, we conclude that the ordered orthodox semigroup W is split.

Now since

$$(e_a, a, f_a) \leqslant (e_b, b, f_b) \Rightarrow a = e_a a f_a \leqslant e_b b f_b = b$$

it is clear that π induces an ordered semigroup isomorphism from $W/\mathscr{Y}$ onto $\operatorname{Im} \pi$; and since S is amenably ordered the assignment $a \Leftrightarrow (e_a, a, f_a)$ clearly yields an ordered semigroup isomorphism from $\operatorname{Im} \pi$ onto S. Thus, as ordered semigroups, $W/\mathscr{Y} \simeq S$.

Finally, let C be the band of idempotents of W and let $\phi: C \to B$ be the algebraic isomorphism (see the proof of Theorem 2.5) given by

$$(e, a, f)\phi = ef.$$

It is clear that ϕ is isotone. Suppose now that $(e, a, f)\phi \leqslant (u, b, v)\phi$. Then $ef \leqslant uv$ and, as in the proof of Theorem 2.5, $ef \mathscr{D} e_a$ and $uv \mathscr{D} e_b$. Since $e_a, e_b \in E$, a skeleton of B, it follows from property (3) in the definition of skeleton in the ordered case that $e_a \leqslant e_b$ whence, since $a^2 = a$ implies $e_a = f_a = a$, we have $a \leqslant b$ and

$$f = f_a f = e_a f = e_a ef \leqslant e_b uv = e_b v = v.$$

Similarly we have $e \leqslant u$. It follows that ϕ is an isomorphism of ordered semigroups. ∎

Our objective now is to obtain an analogue of Theorem 2.8 in the case of ordered semigroups. For this purpose we require the following result.

LEMMA 4.10. *Let T be an orthodox semigroup and let $p, q \in T$ be such that $p \mathscr{Y} q$. Then, for all $p', q' \in V(p) = V(q)$,*

$$pq' = pp'qq'.$$

Proof. Since

$$pq' \cdot qq' = p \cdot q'qq' = pq' \qquad \text{and} \qquad qq' \cdot pq' = q \cdot q'pq' = qq'$$

we have $pq' \mathscr{L} qq'$. Likewise

$$pq' \cdot pp' = pq'p \cdot p' = pp' \qquad \text{and} \qquad pp' \cdot pq' = pp'p \cdot q' = pq'$$

so that $pq' \mathscr{R} pp'$. Since every $\mathscr{D}$-class in a band is rectangular, we conclude that

$$pp' \cdot qq' = pp' \cdot pq' \cdot qq' = pq'. \quad \blacksquare$$

THEOREM 4.11. *Let T be a split amenably ordered orthodox semigroup. If $\pi\colon T/\mathscr{Y} \to T$ is an isotone splitting homomorphism then, with the same notation as in Theorem 2.8, $\operatorname{Im} \pi$ is amenably ordered, θ is balanced and, as ordered semigroups,*

$$T \simeq W(B, \operatorname{Im} \pi, \theta).$$

Proof. Using the same notation as in the proof of Theorem 2.8, we have $\tilde{x} = x \natural \pi$ for every $x \in T$. Since $\natural$, π are isotone it follows that $E = B \cap \operatorname{Im} \pi$ is a skeleton of the ordered band B. Now, as in Theorem 2.8, we have an algebraic isomorphism $\psi\colon W \to T$ given by $(e, a, f)\psi = eaf$. Defining $(e, a, f) \leqslant (u, b, v) \Leftrightarrow e \leqslant u$, $a \leqslant b$, $f \leqslant v$ we see immediately that ψ is isotone. Suppose now that $(e, a, f)\psi \leqslant (u, b, v)\psi$, say

$$x = eaf \leqslant ubv = y.$$

Then since, from Theorem 2.8, $x = (x\tilde{x}^{-1}, \tilde{x}, \tilde{x}^{-1}x)\psi$ we have

$$e = x\tilde{x}^{-1}, \qquad a = \tilde{x}, \qquad f = \tilde{x}^{-1}x, \qquad u = y\tilde{y}^{-1}, \qquad b = \tilde{y}, \qquad v = \tilde{y}^{-1}y.$$

Since $x \leqslant y$ and $\natural\pi$ is isotone we have $a = \tilde{x} \leqslant \tilde{y} = b$; further since T is amenably ordered, $xx' \leqslant yy'$ and $x''x \leqslant y''y$ for some $x', x'' \in V(x)$ and $y', y'' \in V(y)$. Again since $\natural\pi$ is isotone,

$$\tilde{x}\tilde{x}^{-1} = \widetilde{xx'} \leqslant \widetilde{yy'} = \tilde{y}\tilde{y}^{-1}$$

and similarly

$$\tilde{x}^{-1}\tilde{x} = \widetilde{x''x} \leqslant \widetilde{y''y} = \tilde{y}^{-1}\tilde{y}.$$

It now follows by Lemma 4.10 that

$$e = x\tilde{x}^{-1} = xx'\tilde{x}\tilde{x}^{-1} \leqslant yy'\tilde{y}\tilde{y}^{-1} = y\tilde{y}^{-1} = u$$

and similarly that $f \leqslant v$. Thus $(e, a, f) \leqslant (u, b, v)$ and so we see that ψ is also an order isomorphism. Consequently, W is an ordered semigroup with respect to the cartesian order.

Finally, if $a, b \in \operatorname{Im} \pi$ are such that $a \leqslant b$ then

$$aa^{-1} = \tilde{a}\tilde{a}^{-1} \leqslant \tilde{b}\tilde{b}^{-1} = bb^{-1} \qquad \text{and} \qquad a^{-1}a = \tilde{a}^{-1}\tilde{a} \leqslant \tilde{b}^{-1}\tilde{b} = b^{-1}b$$

and so $\operatorname{Im} \pi$ is amenably ordered. It is now a consequence of Theorem 4.6 that the triangulation θ is balanced. $\blacksquare$

As mentioned in Remark 4.8, a distinction must be maintained between the notion of a split ordered orthodox semigroup and that of an ordered split orthodox semigroup. While we have been concerned uniquely with the first of these, we shall now show how they are related. In order to avoid syntactical difficulties in what follows, we shall say that an ordered orthodox semigroup *splits weakly* if it is split algebraically (i.e. in the sense of Definition 1.4) and that it *splits strongly* if it is split in the sense of Definition 4.7.

THEOREM 4.12. *Let T be an amenably ordered orthodox semigroup with band of idempotents B. Then the following statements are equivalent*:

(1) *T splits strongly*;

(2) *T splits weakly and B splits strongly.*

Proof. It is clear that (1) $\Rightarrow$ (2). As for (2) $\Rightarrow$ (1), it suffices to show that the mapping $\natural\pi: x \mapsto \tilde{x}$ is isotone; for then, as in the proof of Theorem 4.11, we have $T \simeq W(B, \operatorname{Im}\pi, \theta)$ as ordered semigroups. Suppose then that $a, b \in T$ are such that $a \leqslant b$. Then there exist $a' \in V(a)$ and $b' \in V(b)$ such that $aa' \leqslant bb'$. Since, by hypothesis, B splits strongly we have $\widetilde{aa'} \leqslant \widetilde{bb'}$; and similarly $\widetilde{a'a} \leqslant \widetilde{b'b}$. But

$$\widetilde{aa'}a\widetilde{a'a} = \tilde{a}\tilde{a}^{-1}a\tilde{a}^{-1}\tilde{a} = \tilde{a}\tilde{a}^{-1}\tilde{a} = \tilde{a}$$

and similarly $\widetilde{bb'}b\widetilde{b'b} = \tilde{b}$. It therefore follows that $\tilde{a} \leqslant \tilde{b}$. ∎

THEOREM 4.13. *Let T be an amenably ordered orthodox semigroup whose B of idempotents is normal. Then T splits strongly if and only if B splits strongly.*

Proof. If B splits strongly then B splits weakly whence, by Theorem 1.8, T splits weakly. The result now follows by Theorem 4.12.

COROLLARY 4.14. *Let T be an amenably ordered orthodox semigroup whose band B of idempotents is normal. If $T/\mathscr{Y}$ has an identity then T splits strongly.*

Proof. Let $\xi \in B$ be such that $\mathscr{Y}_\xi$ is the identity of $T/\mathscr{Y}$. Then, by Corollary 1.9, B splits weakly; and since

$$a \leqslant b \Rightarrow \tilde{a} = \xi a \xi \leqslant \xi b \xi = \tilde{b}$$

we see that B splits strongly. ∎

5. THE PRESENCE OF A GREATEST IDEMPOTENT

We shall now consider ordered orthodox semigroups that have a greatest idempotent and show how splitting is governed by properties of this element. The fundamental observation in this direction is the following.

LEMMA 5.1. *Let S be an ordered regular semigroup with greatest element ξ. Then the subsemigroup $\xi S\xi$ is commutative and idempotent. Moreover, the ordering on S extends the natural ordering on $\xi S\xi$.*

Proof. Let ξ' be an inverse of ξ. Then from $\xi\xi' \leqslant \xi$ we obtain $\xi = \xi\xi'\xi \leqslant \xi^2 \leqslant \xi$ and so $\xi^2 = \xi$. Let $\xi S\xi$; then T is regular and has greatest element ξ that is an identity. Now for $x, y \in T$ we have $xy \leqslant x\xi = x$ and $xy \leqslant \xi y = y$. In particular, if x' is an inverse of x in T then $x = xx'x \leqslant xx' \leqslant x$ so that $x = xx'$. Thus T is a band. Finally, if $z \leqslant x, y$ with $z = z^2$ then $z = z^2 \leqslant xy$, so that $x \wedge y$ exists and coincides with xy. ∎

THEOREM 5.2. *Let B be an ordered band with greatest element ξ. Then $\xi B\xi$ is a semilattice. Moreover, if $B/\mathscr{D}$ is equipped with the natural order then the following statements are equivalent:*

(1) *B is split;*

(2) *$\natural\colon B \to B/\mathscr{D}$ is isotone;*

(3) $(\forall x \in B)\ x = x\xi x$;

(4) $(\forall x, y \in B)\ xy = x\xi y$ *(i.e., ξ is a middle unit);*

(5) *$\xi B\xi$ is a skeleton of B;*

(6) *$\mathscr{D}$ is a closure equivalence with set of closed elements $\xi B\xi$;*

(7) *$\natural\colon B \to B/\mathscr{D}$ is residuated.*

Moreover, when any of these conditions are satisfied B is normal.

Proof. That $\xi B\xi$ is a semilattice is immediate from Lemma 5.1.

(1) ⇒ (2): Clear.

(2) ⇒ (3): Let $B = \bigcup\{B_\alpha;\ \alpha \in Y\}$ where each B_α is a rectangular band. Since ξ is the greatest element of B, it follows by (2) that the $\mathscr{D}$-class of ξ is the greatest $\mathscr{D}$-class of B. As can be seen on passing to quotients modulo $\mathscr{D}$, we then have $x\,\mathscr{D}\,x\xi x$ for every $x \in B$. Since the $\mathscr{D}$-classes are rectangular bands it follows that, for every $x \in B$, $x = x^2 = x \cdot x\xi x \cdot x = x\xi x$.

(3) ⇒ (4): It is clear that the identity $z = z\xi z$ implies that the $\mathscr{D}$-class of ξ is the greatest $\mathscr{D}$-class of B. Consequently, for all $x, y \in B$,

$$\xi x\xi \cdot \xi y\xi\ \mathscr{D}\ \xi xy\xi.$$

But, by Lemma 5.1, $\xi x\xi \cdot \xi y\xi$ and $\xi xy\xi$ commute. Hence we have $\xi x\xi \cdot \xi y\xi = \xi xy\xi$ and so, by (3),

$$\begin{aligned} xy = x\xi x \cdot y\xi y &= x \cdot \xi xy\xi \cdot y \\ &= x \cdot \xi x\xi \cdot \xi y\xi \cdot y \\ &= x\xi x \cdot \xi \cdot y\xi y \\ &= x\xi y, \end{aligned}$$

so that ξ is a middle unit.

(4) $\Rightarrow$ (3): Clear.

(3) $\Rightarrow$ (5): From (3) we have $x \cdot \xi x = x$ and consequently $x \mathscr{L} \xi x$. Moreover, $\xi x \cdot \xi x \xi = \xi x \xi$ and $\xi x \xi \cdot \xi x = \xi x$, so that $\xi x \mathscr{R} \xi x \xi$. Thus $x \mathscr{D} \xi x \xi$ and so $\xi B \xi$ meets every $\mathscr{D}$-class at least once. If now $y, z \in \xi B \xi$ with $y \mathscr{D} z$ then y and z have the same inverses in B and in particular they have the same inverse in the inverse subsemigroup $\xi B \xi$; consequently, $y = z$. Thus $\xi B \xi$ meets every $\mathscr{D}$-class at most once. Since $x \leqslant y$ gives $\xi x \xi \leqslant \xi y \xi$ it now follows that $\xi B \xi$ is a skeleton of B.

(5) $\Rightarrow$ (1): Lemma 4.3.

(3) $\Rightarrow$ (6): If (3) holds then, as in the proof of (3) $\Rightarrow$ (5), we have $x \mathscr{D} \xi x \xi$ for every $x \in B$ with $x = xxx \leqslant \xi x \xi$; moreover $\xi y \xi \mathscr{D} \xi z \xi$ implies that, $\xi y \xi = \xi z \xi$. Writing $\bar{x} = \xi x \xi$ we thus have $x \leqslant \bar{x}$, $\bar{\bar{x}} = \bar{x}$, $x \leqslant y \Rightarrow \bar{x} \leqslant \bar{y}$ $x \mathscr{D} \bar{x}$, $x \mathscr{D} y \Leftrightarrow \bar{x} = \bar{y}$. Consequently, $\mathscr{D}$ is a closure equivalence whose set of closed elements is $\xi B \xi$.

(6) $\Rightarrow$ (7): If now (6) holds then the greatest element in the class of x modulo $\mathscr{D}$ is $\bar{x} = \xi x \xi$. The order induced on $B/\mathscr{D}$ by virtue of the fact that $\mathscr{D}$ is a closure is given by

$$\mathscr{D}_a \leqslant \mathscr{D}_b \Leftrightarrow \bar{a} \leqslant \bar{b}.$$

Now the natural order on $B/\mathscr{D}$ is given by

$$\mathscr{D}_a \leqslant \mathscr{D}_b \Leftrightarrow \mathscr{D}_a = \mathscr{D}_{ab} \Leftrightarrow \bar{a} = \overline{ab} = \bar{a}\bar{b} \Leftrightarrow \bar{a} \leqslant \bar{b}.$$

Since, by Lemma 5.1, the ordering of B extends the natural order on $\xi B \xi$ it follows that these two orderings on $B/\mathscr{D}$ coincide. Consequently, $\natural: B \to B/\mathscr{D}$ is isotone and is residuated since $a\natural \leqslant \mathscr{D}_b \Leftrightarrow a \leqslant \bar{b}$.

(7) $\Rightarrow$ (2): Clear.

If now any of the above conditions are satisfied then for all $x, y, z, w \in B$ we have

$$\begin{aligned} xyzw &= x \cdot \xi y \xi \cdot \xi z \xi \cdot w && \text{by (4)} \\ &= x \cdot \xi z \xi \cdot \xi y \xi \cdot w && \text{by Lemma 5.1} \\ &= xzyw && \text{by (4).} \end{aligned}$$

Hence B is normal. ∎

In view of the previous result, we now consider the case where $\mathscr{Y}$ is a closure equivalence.

THEOREM 5.3. *Let S be an ordered orthodox semigroup. Then the following statements are equivalent:*

(1) *$\mathscr{Y}$ is a closure equivalence;*

(2) *$S/\mathscr{Y}$ can be ordered in such a way $\natural: S \to S/\mathscr{Y}$ is residuated.*

Moreover, if $\mathscr{Y}$ is a multiplicative closure, S splits strongly and every element of S has a greatest inverse.

Proof. (1) $\Rightarrow$ (2): Since every closure equivalence is strongly upper regular it is clear that $S/\mathscr{Y}$ can be ordered by

$$\mathscr{Y}_a \leqslant \mathscr{Y}_b \Leftrightarrow \bar{a} \leqslant \bar{b}$$

where $\bar{x}$ denotes the greatest element of $\mathscr{Y}_x$ for every $x \in S$. The map $\natural\colon S \to S/\mathscr{Y}$ is now clearly isotone with $a\natural \leqslant \mathscr{Y}_b \Leftrightarrow a \leqslant \bar{b}$. Thus $\natural$ is residuated.

(2) $\Rightarrow$ (1): Suppose now that (2) holds. Let $\pi\colon S/\mathscr{Y} \to S$ be the residual of $\natural$. Writing $\tilde{x} = x\natural\pi$ for every $x \in S$ we have $\tilde{x} \geqslant x$ and, since $\natural\pi\natural = \natural$, $\tilde{\tilde{x}} = \tilde{x}$ and $\tilde{x}\natural = x\natural$ so that $\tilde{x}\, \mathscr{Y}\, x$. It follows immediately that $\mathscr{Y}$ is a closure equivalence.

If now $\mathscr{Y}$ is a multiplicative closure then the residual π of $\natural$ is a splitting homomorphism; for $\pi\natural = id_{S/\mathscr{Y}}$. Since $\natural, \pi$ are each isotone it follows that S splits strongly. Moreover, as observed in the proof of Theorem 2.8, $\tilde{x}^{-1} \in V(x)$ for every $x \in T$; and since $\tilde{x}^{-1}$ is the greatest element in its $\mathscr{Y}$-class it follows that $\tilde{x}^{-1}$ is the greatest inverse of x. ▮

THEOREM 5.4. *Let S be an ordered orthodox semigroup with greatest idempotent ξ. Then $\xi S\xi$ is an ordered inverse semigroup of which the identity element ξ is the greatest idempotent. Moreover, the following statements are equivalent:*

(1) *$\mathscr{Y}_\xi$ is the identity of $S/\mathscr{Y}$;*

(2) $(\forall x = x^2 \in S)\ x = x\xi x$;

(3) $e^2 = e \leqslant f = f^2 \Rightarrow e = efe$;

(4) $(\forall x, y \in S)\ xy = x\xi y$;

(5) $(\forall x \in S)\ x^2 = x\xi x$;

(6) *$\xi S\xi$ meets every $\mathscr{Y}$-class of S exactly once;*

(7) *$\mathscr{Y}$ is a multiplicative closure equivalence with set of closed elements $\xi S\xi$.*

Moreover, when any of these conditions are satisfied, S splits strongly, every element of S has a greatest inverse and the band of idempotents of S is normal.

Proof. $\xi S\xi$ is an ordered regular semigroup and, as in Lemma 5.1, its idempotents commute. Hence $\xi S\xi$ is an inverse semigroup.

(1) $\Rightarrow$ (2): If the $\mathscr{Y}$-class of ξ is the identity of $S/\mathscr{Y}$ then it is clear that, for every idempotent $x \in S$, $x\, \mathscr{D}\, x\xi x$. Since the $\mathscr{D}$-classes are rectangular bands we deduce that $x = x\xi x$.

(2) $\Rightarrow$ (3): If (2) holds and $e^2 = e \leqslant f = f^2$ then, since ξ is the greatest idempotent, $e = eee \leqslant efe \leqslant e\xi e = e$ so $e = efe$.

(3) $\Rightarrow$ (2): Immediate from the fact that $e \leqslant \xi$.

(2) $\Rightarrow$ (4): If (2) holds then by Theorem 5.2 we have $e\xi f = ef$ for all idempotents e, f. Thus, if $x, y \in S$ and $x' \in V(x), y' \in V(y)$ then

$$xy = xx'xyy'y = xxx'\xi yy'y = x\xi y.$$

(4) $\Rightarrow$ (5): Clear.
(5) $\Rightarrow$ (2): Clear.
(4) $\Rightarrow$ (6): Given $x \in S$, let $x' \in V(x)$. Since ξ is a middle unit,

$$\xi x\xi \cdot x' \cdot \xi x\xi = \xi xx'x\xi = \xi x\xi;$$
$$x' \cdot \xi x\xi \cdot x' = x'xx' = x'.$$

Thus x and $\xi x\xi$ have an inverse x' in common and so $x\,\mathscr{Y}\,\xi x\xi$. Consequently $\xi S\xi$ meets every $\mathscr{Y}$-class at least once. If now $x, y \in \xi S\xi$ with $x\,\mathscr{Y}\,y$ then x, y have the same set of inverses in S; and in particular they have the same inverse in the inverse subsemigroup $\xi S\xi$. Thus $x = y$ and $\xi S\xi$ meets every $\mathscr{Y}$-class exactly once.

(6) $\Rightarrow$ (7): Given $x \in S$, let $\bar{x}$ be the unique element of $\xi S\xi$ that is $\mathscr{Y}$-equivalent to x. Then $\bar{x} = \xi\bar{x}\xi\,\mathscr{Y}\,\xi x\xi$ gives $\bar{x} = \xi x\xi$. If $x' \in V(x)$ it follows from the fact that ξ is the greatest idempotent that $x = xx'xx'x \leqslant \xi x\xi$. Applying (6) again, we see that $\mathscr{Y}$ is closure equivalence with set of closed elements $\xi S\xi$; moreover, since $\xi S\xi$ is a subsemigroup $\mathscr{Y}$ is a multiplicative closure.

(7) $\Rightarrow$ (2): If (7) holds then, on the band B of idempotents of S, $\mathscr{D}$ is a closure equivalence with set of closed elements $\xi B\xi$. (2) now follows by Theorem 5.2.

The final statements now follow from Theorems 5.3 and 5.2. ∎

We now consider the properties (2) and (5) of Theorem 5.4 in relation to ordered regular semigroups that have a greatest element.

THEOREM 5.5. *Let S be an ordered regular semigroup with greatest element ξ. Then the following conditions on S are equivalent*:

(1) $(\forall x = x^2 \in S)\; x = x\xi x$;

(2) $(\forall x \in S)\; x = x\xi x$.

When these conditions hold, we also have

(3) $(\forall x \in S)\; x^3 = x^2 \leqslant x$.

Moreover, the subsemigroup generated by ξ and x is regular, finite, of order 1, 2, 3, 4, 5, 8 *or* 9, *has least element x^2 and contains at most one non-idempotent* (*namely x*).

Proof. It is clear that (2) $\Rightarrow$ (1). Suppose then that (1) holds. Given $x \in S$, let x' be an inverse of x; then

$$\begin{aligned} x = xx'x &= x \cdot x'x \cdot x'x \\ &= x \cdot x'x\xi x'x \\ &= x\xi x'x \\ &= x\xi x' \cdot xx' \cdot xx' \cdot x \\ &= x\xi x'xx'\xi xx'x \\ &= x\xi x'\xi x. \end{aligned}$$

Consequently, $\xi x'\xi$ is an inverse of $\xi x\xi$ in $\xi S\xi$ which, by Lemma 5.1, is a commutative band. It follows that $\xi x'\xi = \xi x\xi$. Thus

$$x\xi x\xi = x \cdot \xi x\xi \cdot \xi x\xi = x\xi x'\xi x\xi = x\xi$$

and so

$$x = x\xi x'\xi x = x\xi x\xi x = x\xi x.$$

If now (1) or (2) holds then, for all $x \in S$,

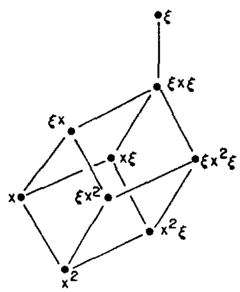
$$x^2 = x^2\xi x^2 = x \cdot x\xi x \cdot x = x^3;$$

$$x = x\xi x \geqslant xxx = x^3,$$

so that (3) holds. Moreover, the subsemigroup generated by ξ and x is readily seen to be

$$X = \langle \xi, x\rangle = \{\xi, x, \xi x, x\xi, \xi x\xi, x^2, \xi x^2, x^2\xi, \xi x^2\xi\}.$$

This is also readily seen to be regular and the only non-idempotent is (possibly) the element x. Providing that no collapsing occurs, the Hasse diagram for X is easily seen to be



An exhaustive study of possible collapsing in this diagram shows that the order of X is either 1, 2, 3, 4, 5, 8 or 9. [For example, if $x = \xi x$ then $x^2 = x\xi x = x$ and the diagram has at most 3 elements; and if $x^2 = x$ then the diagram has at most 5 elements.] We shall show that the maximum order 9 is attainable (in which case x is non-idempotent). Let $\xi\colon \{0, 1,..., 9\} \to \{0, 1,..., 9\}$ be given by

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & 1 & 6 & 7 & 8 & 9 & 6 & 7 & 8 & 9 \end{pmatrix}$$

and let $x\colon \{0, 1,..., 9\} \to \{0, 1,..., 9\}$ be given by

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 2 & 4 & 3 & 3 & 5 & 5 & 2 & 3 & 4 & 5 \end{pmatrix}.$$

Then clearly $\xi^2 = \xi$, $x^2 = x^3$ and $x = x\xi x$. Since 0 has a different image under every element of X, it follows that

$$\langle \xi, x;\ \xi^2 = \xi, x^2 = x^3, x = x\xi x\rangle$$

has order 9, is regular and has only one non-idempotent, namely x. If this semigroup is ordered in such a way that ξ is the greatest element then we have $\xi > x = x\xi x > x^3 = x^2$ from which the above 9-element Hasse diagram results. Moreover, multiplication by ξ and x preserves this order so that we do have an ordered semigroup. ∎

COROLLARY 5.6. *Let S be an ordered regular semigroup with greatest element ξ. If $x^2 = x\xi x$ for every $x \in S$ then S is a normal band and ξ is a middle unit.*

Proof. We have $x = x\xi x$ for every idempotent x so, by Theorem 5.5, S is a band. The result now follows from Theorem 5.2. ∎

At this stage we can now make contact with the results in [2] and [3]; indeed it was these results that gave birth to the notion of splitting.

THEOREM 5.7. *Let S be a Dubreil–Jacotin regular semigroup with bimaximum element ξ. If $x^2 = x\xi x$ for every $x \in S$ then S is orthodox. Moreover, S splits strongly, ξ is a middle unit, the band of idempotents is normal and every element of S has a greatest inverse. Furthermore, if S is amenably ordered then, for every $x \in S$, $x\xi \wedge \xi x$ exists and is x.*

Proof. Let $\zeta\colon S \to G$ be a principal epimorphism and let B be the pre-image under ζ of the identity element of the ordered group G. Then B is a full regular subsemigroup of S, has greatest element ξ and obeys the hypothesis on S. Hence B is a band, namely the band of idempotents of S, so S is orthodox. The remaining statements, save for the last, follow from Theorems 5.2 and 5.4. As for the final statement, we observe from the proof of Theorem 5.4 that the mapping $\alpha\colon S/\mathscr{Y} \to \xi S\xi$ given by $\mathscr{Y}_x\alpha = \xi x\xi$ induces a splitting homomorphism $\beta\colon S/\mathscr{Y} \to S$ with $\operatorname{Im}\beta = \xi S\xi$. By Theorem 4.11, we thus have an ordered semigroup isomorphism

$$S \simeq W = W(B, \xi S\xi, \theta).$$

Now, as is readily seen, the bimaximum element of W is (ξ, ξ, ξ). Moreover, using the fact that $\theta_\xi = id_{\xi B\xi}$ and also the identity $f\xi f = f$ in B, we have

$$\begin{aligned}(e, a, f)(\xi, \xi, \xi) &= (e(f\xi)\,\bar{\theta}_{a^{-1}},\, a,\, (f\xi)\,\bar{\theta}_\xi\xi)\\ &= (e(f_af\xi f_a)\,\theta_{a^{-1}},\, a,\, (\xi f\xi)\,\theta_\xi\xi)\\ &= (e(f_af\xi ff_a)\,\theta_{a^{-1}},\, a,\, \xi f\xi)\\ &= (e(f_aff_a)\,\theta_{a^{-1}},\, a,\, \xi f\xi)\\ &= (e\cdot f_a\theta_{a^{-1}},\, a,\, \xi f\xi)\\ &= (ee_a,\, a,\, \xi f\xi)\\ &= (e, a, \xi f\xi).\end{aligned}$$

In a similar way we can show that

$$(\xi, \xi, \xi)(e, a, f) = (\xi e \xi, a, f).$$

Since $\xi f \xi \geqslant f^3 = f$ it is now clear that $(e, a, f)(\xi, \xi, \xi) \wedge (\xi, \xi, \xi)(e, a, f)$ exists and coincides with (e, a, f). The result now follows from the fact that S is order-isomorphic to W. ▌

6. Concluding Remark

Suppose that S is an orthodox semigroup with band of idempotents B and let E be a $\mathscr{D}$-transversal of B. Then the span $\mathrm{Sp}(E)$ of E meets every $\mathscr{Y}$-class of S exactly once (see Lemma 1.6). As in Section 2, $\mathrm{Sp}(E)$ can be used to give a coordinatisation of S as a set of triples

$$\{(e, a, f) \in B \times \mathrm{Sp}(E) \times B;\ e \mathscr{L} e_a, f \mathscr{R} f_a\}.$$

Based on this, one could construct a Schreier-type structure theory for orthodox semigroups.

In the ordered case, where S is amenably ordered, the proof of Theorem 4.11 shows that this coordinatisation is an order siomorphism provided that E is isotone and the triples are given the cartesian ordering.

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